A CELLULAR DESCRIPTION OF THE DERIVED CATEGORY OF A STRATIFIED SPACE

by

Allen Dudley Shepard

B.A., Hampshire College, 1980

Thesis

Submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in the Department of Mathematics at Brown University

May, 1985

This dissertation by Allen Dudley Shepard

is accepted in its present form by the Department of Mathematics

as satisfying the

dissertation requirement for the degree of Doctor of Philosophy

Date 12 July, 1984 . . Robert D. MacPherson

Recommended to the Graduate Council

Date 7/11/84 . . . . . R. Mark Goresky . . . .

Date July 13, 84 . . . .

Date 7/19/84 . . . . . William Fulton . . . . . .

Approved by the Graduate Council

Date 6 NOV 84 . . . .

VITA

Allen Shepard was born on August 27, 1956 in Boston, Mass. He received a Bachelor of Arts degree from Hampshire College in January of 1980. As a graduate student at Brown University he held various fellowships, teaching assistantships, and research assistantships.

ACKNOWLEDGEMENTS

I would first and foremost like to thank my advisor, Robert MacPherson, whose insights, patience, and guidance have been more than invaluable to me. Thanks also go to Kari Vilonen for the innumerable mathematical discussions we've had and the support he has given me over the years. I am further indebted to Mark Goresky for the discussions we've had and for reading my thesis, as well as to Bill Fulton, who also read this thesis. I would also like to thank Dale Cavanaugh for her excellent typing job.

TABLE OF CONTENTS

INTRODUCTION

In this thesis we develop a concrete and combinatoric theory which parallels that of the derived category of a stratified space, using the data of a regular cell complex structure on the space.

The derived category was first defined by Verdier [V] and was motivated by work of Borel and Moore in 1960 [BM]. It has in recent years come back into focus in a number of branches of mathematics, in particular in the study of topology of singular spaces. In this setting, one generally has a pseudomanifold X with stratification $\mathscr{S}$, and considers the <u>bounded constructible derived category of sheaves</u> on X, $D^b_{\mathscr{S}}(X)$. This is a category whose objects $\underline{\underline{A}}^\bullet$ are chain complexes of sheaves on X with $\underline{\underline{A}}^i = 0$ for $|i| >> 0$ and such that the stalk cohomology sheaves $\underline{\underline{H}}^i\underline{\underline{A}}^\bullet$, when restricted to each stratum, are locally constant with finite dimensional stalks. The morphisms are constructed by a process that causes chain maps that induce isomorphisms between the stalk cohomology sheaves $\underline{\underline{H}}^i\underline{\underline{A}}^\bullet$ to become isomorphisms in the category. The elements of $D^b_{\mathscr{S}}(X)$ naturally occur when studying singular spaces, and in some sense play the same role that locally constant sheaves play in the study of non-singular spaces.

The purpose of this thesis is to study a derived category that is constructed out of <u>cellular sheaves</u> rather than ordinary sheaves. Given a finite regular CW complex structure X on a topological space

$|X|$ (e.g., a simplicial complex), a <u>cellular sheaf</u> $\underline{\underline{S}}$ on X associates to each cell σ a vector space $\underline{\underline{S}}(\sigma)$ and to each face relation $\sigma < \tau$ a linear map $p^{\underline{\underline{S}}}_{\sigma,\tau} : \underline{\underline{S}}(\sigma) \to \underline{\underline{S}}(\tau)$ such that for $\gamma < \sigma < \tau$, $p^{\underline{\underline{S}}}_{\sigma,\tau} \circ p^{\underline{\underline{S}}}_{\gamma,\sigma} = p^{\underline{\underline{S}}}_{\gamma,\tau}$. If the cell complex structure is subordinate to a stratification $|\mathscr{S}|$ of $|X|$, then we can form the bounded constructible cellular derived category $D^b_{\mathscr{S}}(X)$ (objects being chain complexes of cellular sheaves) in the same way that we formed a derived category from ordinary sheaves. The motivation for doing this is given by theorem 5.2.4, which states that $D^b_{\mathscr{S}}(X)$ and $D^b_{|\mathscr{S}|}(|X|)$ are, in a natural way, equivalent as categories. Not only does this give a much more concrete way of describing the elements of $D^b_{|\mathscr{S}|}(|X|)$, but it allows us to use the machinery of cellular sheaves rather than that of ordinary sheaves in studying the derived category, a theory which turns out to be far more concrete and geometric in nature. In the first four chapters, we give a completely self-contained development of cellular sheaf theory and the cellular derived category.

The ideas in this thesis were first developed by Fulton, Goresky, MacPherson, and McCrory in a seminar at Brown University in 1977-78. At that time, most of the content of chapter 1 was worked out for a finite simplicial complex, and the equivalence of the standard and simplicial derived categories was conjectured.

Chapter 1 is concerned with the basic definitions and properties of cell complexes, cellular sheaves, and cellular maps. The exception is §1.2 which develops a purely algebraic theory of n-complexes, the n-dimensional analogues of chain complexes ($n = 1$) or double complexes ($n = 2$). The techniques developed in this section allow us to completely avoid the use of spectral sequences in the paper, and are particularly helpful in chapters 3 and 4. One of the major advantages of the cellular theory is seen in §1.3, where injective and projective cellular sheaves are studied. These sheaves turn out to be extremely simple, as well as plentiful, and it is for these reasons that the cellular theory is as concrete and geometric as it is. This constrasts sharply with the standard theory of sheaves, where there are not enough projectives, and injectives are too unwieldy to be useful except in a purely category theoretic sense.

Chapter 2 gives a development of the homotopy and derived categories of cellular sheaves. Although done in the cellular setting, little of the nature of cellular sheaves is used, and the exposition could be applied to the construction of the derived category of any abelian category.

Chapter 3 is where most of the standard identities in the derived category are proved. Although these statements follow from the equivalence of the two categories and from the similar results in the standard derived category, the arguments used are of a completely different nature, relying on the combinatoric and geometric structures of the cellular category. Usually, proofs of identities are shown by constructing explicit chain maps between complexes, and do not rely on general properties of derived categories.

Chapter 4 develops the theory of triangles in $D^b(X)$, and, similarly to Chapter 2, is essentially a general exposition on triangulated categories applied to the cellular category. One novel feature of this section is the use of the techniques of §1.2 in some of the proofs of standard theorems.

In Chapter 5 an explicit equivalence is given between the standard and cellular derived categories (theorem 5.2.3 and corollary 5.2.4). The exposition is no longer self-contained, as some knowledge of the standard theory of sheaves and derived categories is assumed (for the most part, though, the facts that are assumed are the exact counterparts to the statements proved in the first four chapters in the cellular setting). All functors defined in the thesis are shown to correspond to the standard functor with the same name under this equivalence.

It should be noted that results similar to corollary 5.2.4 were acquired independently by Kashiwara in [K].

CHAPTER ONE

CELLULAR SHEAVES

1.1. Cell Complexes and Cellular Sheaves

Definition: A cell complex X consists of a topological space $|X|$ and a decomposition of $|X|$ into disjoint open topological cells $\{\sigma_i\} = X$ such that if $|X| \cup \{p\}$ is the one-point compactification of $|X|$, then $\{\sigma_i\} \cup \{\{p\}\}$ are the cells of a regular CW complex structure on $|X| \cup \{p\}$.

By a regular CW complex we mean one for which the attaching maps are embeddings. Note that if $|X|$ is compact, a cell complex is simply a regular CW complex. For a subset of cells $A \subseteq X$, $|A| \subseteq |X|$ will denote their union. Also, we will generally write $|\sigma|$ instead of σ when a cell is considered as a topological space, and not just as an element of X.

Lemma 1.1.1: If σ and τ are cells of a cell complex X, $\dim\tau < \dim\sigma$, then either $\overline{|\sigma|} \cap |\tau| = \emptyset$ or $|\tau| \subseteq \overline{|\sigma|}$.

Proof: See R.R.1, section VIII.4, in [CF].

If $|\tau| \subseteq \overline{|\sigma|}$, then τ is called a face of σ, denoted by $\tau \leq \sigma$, or $\tau <_i \sigma$ if $\dim\tau = \dim\sigma - i$. $\overline{\sigma} \subseteq X$ will refer to the set of faces of σ; clearly, $\overline{|\sigma|} = |\overline{\sigma}|$. If $\tau \leq \sigma$ and $\tau \neq \sigma$, τ is called a proper face and is written $\tau < \sigma$. A cell written as σ^p will always be assumed to be of dimension p. We will assume that the cells of X are oriented and for $\tau <_1 \sigma$, will define $[\sigma:\tau]$ to be $+1$ if orientations agree and -1 if orientations disagree.

Lemma 1.1.2: If $\sigma <_2 \tau$, then there are exactly two cells γ_1, γ_2 where $\sigma <_1 \gamma_i <_1 \tau$. We then have $[\gamma_1 : \sigma][\tau : \gamma_1] + [\gamma_2 : \sigma][\tau : \gamma_2] = 0$.

Proof: For the first statement, see R.R.2, section II.1, in [CF] (proof is in section VIII.4). The second statement just says that the chain complex computing CW homology is in fact a chain complex.

Throughout this paper, all vector spaces will be over $\mathbb{Q}$.

Definition: A cellular sheaf $\underline{\underline{S}}$ on X (or simply a sheaf when no confusion can arise) associates a vector space $\underline{\underline{S}}(\sigma)$ to each cell $\sigma \in X$, and a linear "corestriction" map $p^{\underline{\underline{S}}}_{\sigma,\tau} : \underline{\underline{S}}(\sigma) \to \underline{\underline{S}}(\tau)$ (or simply $p_{\sigma,\tau}$ if no confusion can arise) for each pair of cells $\sigma < \tau$ such that $p_{\sigma,\tau} p_{\gamma,\sigma} = p_{\gamma,\tau}$. For convenience, we define $p_{\sigma,\sigma} : \underline{\underline{S}}(\sigma) \to \underline{\underline{S}}(\sigma)$ to be the identity always.

Given a cellular sheaf $\underline{\underline{S}}$, let $C^p_c(X,\underline{\underline{S}}) = \bigoplus_{\sigma^p \in X} \underline{\underline{S}}(\sigma^p)$, and let $\delta_c : C^p_c(X,\underline{\underline{S}}) \to C^{p+1}_c(X,\underline{\underline{S}})$ be given by $\delta_c(f)(\tau) = \sum_{\sigma <_1 \tau} [\tau:\sigma] p_{\sigma,\tau} f(\sigma)$.
Note that $X' = \{\sigma \in X \,|\, |\overline{\sigma}|$ is compact$\}$ is a compact cell complex, and define $C^p(X,\underline{\underline{S}}) = C^p_c(X',\underline{\underline{S}})$ and $\delta : C^p(X,\underline{\underline{S}}) \to C^{p+1}(X,\underline{\underline{S}})$ to be δ_c.

Theorem 1.1.3: $(C^\bullet(X,\underline{\underline{S}}),\delta)$ and $(C^\bullet_c(X,\underline{\underline{S}}),\delta_c)$ are chain complexes.

Proof: For $C^\bullet_c(X,\underline{\underline{S}})$, we have

$$\delta_c\delta_c(f)(\tau) = \sum_{\gamma<_1\tau}[\tau:\gamma]p_{\gamma,\tau}\delta_c f(\gamma)$$

$$= \sum_{\gamma<_1\tau}[\tau:\gamma]p_{\gamma,\tau}\left(\sum_{\sigma<_1\gamma}[\gamma:\sigma]p_{\sigma,\gamma}f(\sigma)\right)$$

$$= \sum_{\sigma<_1\gamma<_1\tau}[\tau:\gamma][\gamma:\sigma]p_{\sigma,\tau}f(\sigma)$$

$$= \sum_{\sigma<_2\tau}([\tau:\gamma_1][\gamma_1:\sigma] + [\tau:\gamma_2][\gamma_2:\sigma])p_{\sigma,\tau}f(\sigma) \quad \text{(where}$$

γ_1 and γ_2 are as in Lemma 1.1.2)

$$= 0 .$$

The result for $C^{\bullet}(X,\underline{\underline{S}})$ follows from this.

<u>Definition</u>: $H^{\bullet}(C^{\bullet}(X,\underline{\underline{S}}),\delta) = H^{\bullet}(X,\underline{\underline{S}})$ is called the <u>cohomology</u> of $\underline{\underline{S}}$, and $H^{\bullet}(C_c^{\bullet}(X,\underline{\underline{S}}),\delta_c) = H_c^{\bullet}(X,\underline{\underline{S}})$ is called the <u>cohomology with compact support</u> of $\underline{\underline{S}}$.

<u>General Definitions and Remarks</u>

A <u>map</u> between (cellular) sheaves $r : \underline{\underline{S}} \to \underline{\underline{T}}$ over a cell complex X is a collection of linear maps $r(\sigma) : \underline{\underline{S}}(\sigma) \to \underline{\underline{T}}(\sigma)$ such that $p^{\underline{\underline{T}}}_{\sigma,\tau}r(\sigma) = r(\sigma)p^{\underline{\underline{S}}}_{\sigma,\tau}$ $\forall\sigma < \tau$.

The direct sum of two sheaves $\underline{\underline{S}} \oplus \underline{\underline{T}}$ over X can be formed in a natural way: $(\underline{\underline{S}} \oplus \underline{\underline{T}})(\sigma) = \underline{\underline{S}}(\sigma) \oplus \underline{\underline{T}}(\sigma)$, $p^{\underline{\underline{S}}\oplus\underline{\underline{T}}}_{\sigma,\tau} = p^{\underline{\underline{S}}}_{\sigma,\tau} \oplus p^{\underline{\underline{T}}}_{\sigma,\tau}$. A map $f : \bigoplus_i \underline{\underline{S}}_i \to \bigoplus_j \underline{\underline{T}}_j$ is given exactly by specifying all the component maps $f_{ij} : \underline{\underline{S}}_i \to \underline{\underline{T}}_j$. Also, it is easily verified that $H^p_{(c)}(X,\underline{\underline{S}} \oplus \underline{\underline{T}}) =$

$H^p_{(c)}(X,\underline{\underline{S}}) \oplus H^p_{(c)}(X,\underline{\underline{T}})$.

Given a map $r : \underline{\underline{S}} \to \underline{\underline{T}}$ of sheaves on X, we can define the sheaf $\underline{\underline{Ker}}(r)$ on X by $\underline{\underline{Ker}}(r)(\sigma) = \mathrm{Ker}(r(\sigma))$, $p^{\underline{\underline{Ker}}(r)}_{\sigma,\tau} = p^{\underline{\underline{S}}}_{\sigma,\tau}\Big|_{\underline{\underline{Ker}}(r)(\sigma)}$. In a similar manner, we can define the sheaf $\underline{\underline{Im}}(r)$, a subsheaf of $\underline{\underline{T}}$, where $\underline{\underline{Im}}(r)(\sigma) = \mathrm{Image}(r(\sigma))$, and the sheaf $\underline{\underline{Cok}}(r)$ where $\underline{\underline{Cok}}(r)(\sigma) = \mathrm{Cokernel}(r(\sigma))$.

The tensor product of two sheaves $S \otimes T$ is formed by $S \otimes T(\sigma) = S(\sigma) \otimes T(\sigma)$ and $p^{\underline{\underline{S}}\otimes\underline{\underline{T}}}_{\sigma,\tau} = p^{\underline{\underline{S}}}_{\sigma,\tau} \otimes p^{\underline{\underline{T}}}_{\sigma,\tau}$.

Let $\cdots \to \underline{\underline{S}}_i \xrightarrow{d^i} \underline{\underline{S}}^{i+1} \xrightarrow{d^{k+1}} \underline{\underline{S}}^{i+2} \to \cdots$ be a chain complex of sheaves. Notice that $(\underline{\underline{S}}^\bullet, d^\bullet)$ can also be thought of as a structure that associates to each cell σ a chain complex of vector spaces $(\underline{\underline{S}}^\bullet(\sigma), d^\bullet(\sigma))$ and to each face relation $\sigma < \tau$ a chain map $p^\bullet_{\sigma,\tau} : (\underline{\underline{S}}^\bullet(\sigma), d^\bullet(\sigma)) \to (\underline{\underline{S}}^\bullet(\tau), d^\bullet(\tau))$ such that $p^\bullet_{\sigma,\tau} p^\bullet_{\gamma,\sigma} = p^\bullet_{\gamma,\tau}$. The i^{th} <u>cohomology sheaf</u> $\underline{\underline{H}}^i(\underline{\underline{S}}^\bullet)$ (or $\underline{\underline{H}}^i(X,\underline{\underline{S}}^\bullet)$) is defined by $\underline{\underline{H}}^i(\underline{\underline{S}}^\bullet)(\sigma) = H^i(\underline{\underline{S}}^\bullet(\sigma), d^\bullet(\sigma))$, the corestriction maps $d^{\underline{\underline{H}}^i(\underline{\underline{S}}^\bullet)}_{\sigma,\tau}$ being given by the i^{th} induced maps $(p^{\underline{\underline{S}}^\bullet}_{\sigma,\tau})^i_*$ on cohomology.

If $\underline{\underline{A}}^\bullet$ is a chain complex of sheaves, then $\underline{\underline{A}}^\bullet[n]$ is the chain complex where $(\underline{\underline{A}}^\bullet[n])^m = \underline{\underline{A}}^{n+m}$.

If $f^\bullet : A^\bullet \to B^\bullet$ is a chain map between complexes of sheaves that induces isomorphisms on the cohomology sheaves, then $f^\bullet$ is called a <u>quasi-isomorphism</u>. The definition of a quasi-isomorphism between complexes of vector spaces is similar. Clearly, $f^{\bullet}$ is a quasi-isomorphism of complexes of sheaves if and only if $f^{\bullet}(\sigma)$ is a quasi-isomorphism of complexes of vector spaces for every $\sigma \in X$.

§1.2 Hypercohomology

Definition: An n-complex of vector spaces $V^{(n)}$ is a collection of vector spaces $\{V^{\alpha} | \alpha \in \mathbb{Z}^n\}$ and linear maps

$$\{d_i^{(\alpha_1,\ldots,\alpha_i,\ldots,\alpha_n)} : V^{(\alpha_1,\ldots,\alpha_i,\ldots,\alpha_n)} \to V^{(\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_n)} \,\Big|\, (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n \;,\; 1 \le i \le n\} \;,$$

such that $d_i^{(\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_n)} \circ d_i^{(\alpha_1,\ldots,\alpha_i,\ldots,\alpha_n)} = 0$, and each square of maps commutes.

The associated single complex $V^{\bullet}$ of $V^{(n)}$ is the chain complex where $V^j = \prod_{\Sigma\alpha_i = j} V^{(\alpha_1,\ldots,\alpha_n)}$ and the maps between components of V^j and V^{j+1} are all maps $\{\pm d_i^{(\alpha_1,\ldots,\alpha_n)} \,|\, \sum_k \alpha_k = j\}$, the signs being assigned to the d_i^{α}'s in a fixed manner that causes every square of maps in $V^{(n)}$ to anticommute. The fact that $V^{\bullet}$ is a chain complex given such a sign convention is clear. Ways of choosing the signs are easily found; for example, we can assign to $d_k^{(\alpha_1,\ldots,\alpha_n)}$ the sign $(-1)^{\alpha_1+\ldots+\alpha_k}$. This convention will be called the standard sign convention.

Theorem 1.2.1: Let $\{\lambda_i^\beta\}$ and $\{\mu_i^\beta\}$, λ_i^β and $\mu_i^\beta = \pm 1$, $\beta \in \mathbb{Z}^n$, $1 \le i \le n$, be two sign conventions making squares in an n-complex anticommute. Given $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n$, choose a path $d_{i_1}^{\beta 1}$, $d_{i_2}^{\beta 2}, \ldots ,d_{i_k}^{\beta k}$ from $(0,\ldots,0)$ to α in $\mathbb{Z}^n$, each step of the path either increasing or decreasing a single coordinate by 1 (the path does not necessarily go in the same direction as the maps!). Define r^α to be $\prod_{j=1}^{k} \mu_{i_j}^{\beta_j}\lambda_{i_j}^{\beta_j}$. Then,

(i) r^α is independent of the path chosen from 0 to α ;

(ii) if $V^{(n)}$ is an n-complex, $V_\mu^\bullet$ the associated single complex with sign convention μ_i^β , and $V_\lambda^\bullet$ defined similarly, then the map $r^\bullet : V_\mu^\bullet \to V_\lambda^\bullet$ which maps V^α to V^α by multiplying by r^α is an isomorphism of chain complexes.

Proof: (i) It suffices to show that $r^{(0,\ldots,0)} = +1$ for all paths. Let β_1,β_2,β_3 be a sequence of three points in the path. If $\beta_1 = \beta_3$, then it is easy to show that the path acquired by removing the two steps β_1 to β_2 to β_3 has the same value of r^0 . If $\beta_1 \neq \beta_3$, then β_1,β_2,β_3 are three of the corners of a square in $\mathbb{Z}^n$ with fourth corner β_4. It follows from the anticommutativity condition that the path which replaces the steps β_1 to β_2 to β_3 with β_1 to β_4 to β_3 has the same value of r^0 .

Now, find a point $\gamma = (\gamma_1,\ldots,\gamma_n)$ in the path with $\sum_i |\gamma_i|$ maximal, and let β,β' be the points preceding and following γ , respectively. We can apply one of the above two processes to get a new path with the same value of r^0 , but with one less point with maximal value of $\sum_i|\gamma_i|$ (or, if there was a unique maximal value of $\sum_i|\gamma_i|$, a path with smaller maximal value of $\sum_i|\gamma_i|$). Continuing in this manner, the path is eventually reduced to the path of length 0 without the value of r^0 ever changing, so the result follows.

(ii) To show that $r^\bullet$ is a chain map, we want to show that for $\alpha = (\alpha_1,\ldots,\alpha_n)$, $\alpha' = (\alpha_1,\ldots,\alpha_j+1,\ldots,\alpha_n)$, we have $r^\alpha\lambda_j^\alpha = r^{\alpha'}\mu_j^\alpha$ (it is obvious that $r^\bullet$ is an isomorphism, once it is shown to be a chain map).

Let $p_i^{\beta_1},\ldots,p_{i_k}^{\beta_k}$ be a path from 0 to α .

Then $r^\alpha\lambda_j^\alpha = \lambda_{i_1}^{\beta_1}\mu_{i_1}^{\beta_1}\cdots\lambda_{i_k}^{\beta_k}\mu_{i_k}^{\beta_k}\lambda_j^\alpha$

$= \lambda_{i_1}^{\beta_1}\mu_{i_1}^{\beta_1}\cdots\lambda_{i_k}^{\beta_k}\mu_{i_k}^{\beta_k}\lambda_j^\alpha\mu_j^\alpha\mu_j^\alpha = r^{\alpha'}\mu_j^\alpha$ (using the path $p_{i_1}^{\beta_1},\ldots,p_{i_k}^{\beta_k},p_j^\alpha$ from 0 to α').

Theorem 1.2.1 shows that the associated single complex is defined up to isomorphism.

<u>Definition</u>: <u>The hypercohomology</u> of $V^{(n)}$, $\mathbb{H}^\bullet(V^{(n)})$, is the cohomology of the associated single complex.

Given a map $f : V^{(n)} \to W^{(n)}$ between n-complexes, i.e., a collection of maps $f^\alpha : V^\alpha \to W^\alpha$, $\alpha \in \mathbb{Z}^n$, that commute with boundary maps, the f^α will still commute with boundary maps after signs are changed in $V^{(n)}$ and $W^{(n)}$ to make their squares anticommute, and hence f induces maps $f^{\bullet} : V^{\bullet} \to W^{\bullet}$ and $\underline{f}^{\bullet} : \mathbf{H}^{\bullet}(V^{(n)}) \to \mathbf{H}^{\bullet}(W^{(n)})$. If a different choice of sign changes were used, then the induced maps will commute with the canonical isomorphisms in theorem 1.2.1.

Let $V^{(n)}$ be an n-complex and choose r of the n coordinates in $\mathbf{Z}^n$ (we will assume the first r are chosen and will write the coordinates as $(\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s)$, $r + s = n$). We can define a generalization of the associated single complex, called a <u>reduced</u> complex (in this case an (s+1)-complex) by forming the associated single complex of each r complex consisting of vector spaces $V^{(\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s)}$ with $(\beta_1,\ldots,\beta_s)$ fixed. The above process will be referred to as "reducing $V^{(n)}$ in the first r coordinates". The resulting (s+1)-complex will also be referred to as a "partial reduction" of $V^{(n)}$. Note that $V^{\bullet}$ is acquired by reducing $V^{(n)}$ in all coordinates.

Suppose that an n-complex $V^{(n)}$ is reduced to a single complex $\tilde{V}^{\bullet}$ by a sequence of partial reductions. Then $\tilde{V}^k = \bigoplus_{\Sigma\alpha_j = k} V^{(\alpha_1,\ldots,\alpha_n)}$ and the component maps of $\tilde{V}^k \to \tilde{V}^{k+1}$ are $\{\pm d_i^{(\alpha_1,\ldots,\alpha_n)}\Big|_{\Sigma_j \alpha_j = k}$, the signs given by some convention λ_i^{α} . Since this process of reductions always results in a chain complex $\tilde{V}^{\bullet}$, the signs λ_i^{α} applied to $V^{(n)}$ must make each square anticommute. Hence we have the following.

<u>Theorem 1.2.2</u>: If an n-complex $V^{(n)}$ is reduced to a single complex by a sequence of partial reductions, the result is the as-

sociated single complex of $V^{(n)}$ with respect to some sign convention λ_i^α. The sign convention depends only on the sequence of reductions used, and not on the n-complex started with.

Definition: A map between n-complexes that induces isomorphisms on hypercohomology is called a quasi-isomorphism.

Given a map $f^{(n)} : V^{(n)} \to W^{(n)}$ between n-complexes and a fixed sequence of reductions on $V^{(n)}$ and $W^{(n)}$ resulting in s-complexes $V^{(s)}$ and $W^{(s)}$, $f^{(n)}$ induces a map $f^{(s)} : V^{(s)} \to W^{(s)}$ in an obvious manner. It follows from theorem 1.2.2 that $f^{(n)}$ is a quasi-isomorphism if and only if $f^{(s)}$ is.

Definition: A n-complex $V^{(n)}$ is bounded if there exists an N such that $V^{(\alpha_1,\ldots,\alpha_n)} = 0$ whenever $|\alpha_i| > N$ for some i.

Let $f^{(n)} : V^{(n)} \to W^{(n)}$ be a map between bounded n-complexes, and choose r coordinates in $\mathbb{Z}^n$. By fixing the other $n-r$ coordinates in at $(a_{r+1},\ldots,a_n) = a \in \mathbb{Z}^{n-r}$, we get "slices" $V_a^{(r)}$ and $W_a^{(r)}$ of $V^{(n)}$ and $W^{(n)}$, and a map $f_a^{(r)} : V_a^{(r)} \to W_a^{(r)}$ between r-complexes by restricting $f^{(n)}$. We then have,

Theorem 1.2.3: If $f_a^{(r)}$ is a quasi-isomorphism for all $a \in \mathbb{Z}^{n-r}$, then $f^{(n)}$ is a quasi-isomorphism.

Proof: By reducing along the chosen r coordinates we get an induced map $f^{(n-r+1)}$ which is a quasi-isomorphism if and only if $f^{(n)}$ is a quasi-isomorphism. But for each way of fixing the $n-r$

coordinates that were not reduced, $f^{(n-r+1)}$ restricts to a quasi-isomorphism between single complexes, so it is necessary only to show the case $r = 1$.

We will first prove the lemma for the case of a double complex $(n = 2)$. Let $f^{\bullet\bullet} : A^{\bullet\bullet} \to B^{\bullet\bullet}$ be a map between double complexes such that for each j , $f^{\bullet\bullet}$ restricted to the $j^{\underline{th}}$ column, i.e., $f^{\bullet j} : A^{\bullet j} \to B^{\bullet j}$, is a quasi-isomorphism. Let t be the smallest integer such that either A^{vt} or $B^{vt} \neq 0$ for some v and let u be the largest such integer. Define $\tilde{A}^{\bullet\bullet}$ by $\tilde{A}^{ij} = A^{ij}$ for $j > t$, $\tilde{A}^{ij} = 0$ for $j \leq t$, and all boundary maps are the same for $j > t$. Define $\overset{\approx}{A}^{\bullet\bullet}$ to be $\overset{\approx}{A}^{ij} = 0$ for $j \neq t$ and $\overset{\approx}{A}^{it} = A^{it}$, all boundary maps the same for $j = t$. We then have a short exact sequence of double complexes $0 \to \tilde{A}^{\bullet\bullet} \to A^{\bullet\bullet} \to \overset{\approx}{A}^{\bullet\bullet} \to 0$, which gives rise to maps between associated single complexes $0 \to \tilde{A}^{\bullet} \to A^{\bullet} \to \overset{\approx}{A}^{\bullet} \to 0$. It is easily checked that this is a short exact sequence of chain complexes. We also have a similar short exact sequence $0 \to \tilde{B}^{\bullet} \to B^{\bullet} \to \overset{\approx}{B} \to 0$, and $f^{\bullet\bullet}$ induces maps between these. This gives rise to the following commutative diagram, the horizontal sequences exact.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{H}^{p-1}(\overset{\approx}{A}^{\bullet\bullet}) \arrow[r] \arrow[d] & \mathbb{H}^{p}(\tilde{A}^{\bullet\bullet}) \arrow[r] \arrow[d] & \mathbb{H}^{p}(A^{\bullet\bullet}) \arrow[r] \arrow[d] & \mathbb{H}^{p}(\overset{\approx}{A}^{\bullet\bullet}) \arrow[r] \arrow[d] & \mathbb{H}^{p+1}(\tilde{A}^{\bullet\bullet}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbb{H}^{p-1}(\tilde{B}^{\bullet\bullet}) \arrow[r] & \mathbb{H}^{p}(\overset{\approx}{B}^{\bullet\bullet}) \arrow[r] & \mathbb{H}^{p}(B^{\bullet\bullet}) \arrow[r] & \mathbb{H}^{p}(\overset{\approx}{B}^{\bullet\bullet}) \arrow[r] & \mathbb{H}^{p+1}(\tilde{B}^{\bullet\bullet}) \arrow[r] & \cdots
\end{tikzcd}$$

We can now apply induction on $u-t$. Suppose $u-t = n$ for

$f^{\bullet\bullet} : A^{\bullet\bullet} \to B^{\bullet\bullet}$, and we assume that for any map $g^{\bullet\bullet} : C^{\bullet\bullet} \to D^{\bullet\bullet}$ between bounded double complexes, if $g^{\bullet j} : C^{\bullet j} \to D^{\bullet j}$ is a quasi-isomorphism for all j , and $u-t < n$, then $g^{\bullet\bullet}$ is a quasi-isomorphism. Then $f^{\bullet\bullet}$ is a quasi-isomorphism by applying the five-lemma to the diagram above.

To prove the theorem for n-complexes, $n > 2$, use induction on n . Let $f^{(n)}: \mathcal{V}^{(n)} \to \mathcal{W}^{(n)}$ be a quasi-isomorphism on each single complex acquired by fixing the last $n-1$ coordinates. Using the result for $n = 2$, $f^{(n)}$ restricts to a quasi-isomorphism on each double complex acquired by fixing the last $n-2$ coordinates. By reducing $\mathcal{V}^{(n)}$ and $\mathcal{W}^{(n)}$ in the first two coordinates, a map between (n-1)-complexes $f^{(n-1)} : \mathcal{V}^{(n-1)} \to \mathcal{W}^{(n-1)}$ is induced, and $f^{(n-1)}$ restricts to a quasi-isomorphism on each single complex with the last $n-1$ coordinates fixed. By induction, $f^{(n-1)}$ is a quasi-isomorphism, and hence $f^{(n)}$ is a quasi-isomorphism.

As a special case of this, we have,

<u>Corollary 1.2.4</u>: Let $V^{\bullet\bullet}$ be a bounded double complex and $A^{\bullet}$ a bounded single complex mapping into $V^{\bullet\bullet}$ so as to form the following commutative diagram:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
A^0 \arrow[r, hook, "f^0"] \arrow[d] & V^{00} \arrow[r] \arrow[d] & V^{01} \arrow[r] \arrow[d] & V^{02} \arrow[r] \arrow[d] & \cdots \\
A^1 \arrow[r, hook, "f^1"] \arrow[d] & V^{10} \arrow[r] \arrow[d] & V^{11} \arrow[r] \arrow[d] & V^{12} \arrow[r] \arrow[d] & \cdots \\
A^2 \arrow[r, hook, "f^2"] \arrow[d] & V^{20} \arrow[r] \arrow[d] & V^{21} \arrow[r] \arrow[d] & V^{22} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

If each horizontal sequence is exact, then $f^\bullet$ induces an isomorphism $\underline{f}^p : H^p(A^\bullet) \to \mathbb{H}^p(V^{\bullet\bullet})$ for each p.

Proof: This follows immediately from theorem 1.2.2 since $H^\bullet(A^\bullet) = \mathbb{H}^\bullet(A^\bullet \to 0^\bullet \to 0^\bullet \to \cdots)$, $A^\bullet \to 0^\bullet \to 0^\bullet \to \cdots$ being the double complex which has $A^\bullet$ in the 0^{th} column and 0 elsewhere, and $f^\bullet$ extends to a map $f^{\bullet\bullet} : (A^\bullet \to 0^\bullet \to 0^\bullet \to \cdots) \to V^{\bullet\bullet}$ which is a quasi-isomorphism between each pair of horizontal complexes.

Let $(\underline{\underline{S}}^\bullet, d)$ be a complex of sheaves on a cell complex X. Then $(C^i(X,\underline{\underline{S}}^j), d, \delta)$ and $(C_c^i(X,\underline{\underline{S}}^j), d_c, \delta_c)$ are double complexes of vector spaces. Define the *hypercohomology* $\mathbb{H}^\bullet(X,\underline{\underline{S}}^\bullet)$ of $\underline{\underline{S}}^\bullet$ to be $\mathbb{H}^\bullet(C^i(X,\underline{\underline{S}}^j))$ and the *hypercohomology with compact support* $\mathbb{H}_c^\bullet(X,\underline{\underline{S}}^\bullet)$ of $\underline{\underline{S}}^\bullet$ to be $\mathbb{H}^\bullet(C_c^i(X,\underline{\underline{S}}^j))$.

A sheaf $\underline{\underline{S}}$ will be identified with the complex of sheaves $\cdots \to 0 \to \underline{\underline{S}} \to 0 \to 0 \to \cdots$, $\underline{\underline{S}}$ being in degree 0. Under this identification, $\mathbb{H}^{\bullet}_{(c)}(X,\underline{\underline{S}}) = H^{\bullet}_{(c)}(X,\underline{\underline{S}})$.

Theorem 1.2.2 gives us two immediate statements about hypercohomology of complexes of sheaves.

Theorem 1.2.5: (i) If $r^{\bullet} : \underline{\underline{S}}^{\bullet} \to \underline{\underline{T}}^{\bullet}$ is a chain map between bounded complexes of sheaves and induces isomorphisms on the cohomology sheaves $\underline{\underline{r}}^{\bullet} : \underline{\underline{H}}^{\bullet}(X,\underline{\underline{S}}^{\bullet}) \xrightarrow{\cong} \underline{\underline{H}}^{\bullet}(X,\underline{\underline{T}}^{\bullet})$, then

$$\underline{r}^{\bullet} : \mathbb{H}^{\bullet}(X,\underline{\underline{S}}^{\bullet}) \to \mathbb{H}^{\bullet}(X,\underline{\underline{T}}^{\bullet}) \quad \text{and}$$
$$\underline{r}^{\bullet} : \mathbb{H}^{\bullet}_{c}(X,\underline{\underline{S}}^{\bullet}) \to \mathbb{H}^{\bullet}_{c}(X,\underline{\underline{T}}^{\bullet}) \quad \text{are isomorphisms.}$$

(ii) if $r^{\bullet} : \underline{\underline{S}}^{\bullet} \to \underline{\underline{T}}^{\bullet}$ is a chain map between bounded complexes of sheaves and $r^p : \underline{\underline{S}}^p \to \underline{\underline{T}}^p$ induces isomorphisms on the sheaf cohomology (respectively sheaf cohomology with compact support) for all p, then $r^{\bullet}$ induces isomorphisms $\underline{r}^{\bullet} : \mathbb{H}^{\bullet}(X,\underline{\underline{S}}^{\bullet}) \xrightarrow{\cong} \mathbb{H}^{\bullet}(X,\underline{\underline{T}}^{\bullet})$ (respectively $\underline{r}^{\bullet} : \mathbb{H}^{\bullet}_{c}(X,\underline{\underline{S}}^{\bullet}) \xrightarrow{\cong} \mathbb{H}^{\bullet}_{c}(X,\underline{\underline{T}}^{\bullet})$).

Proof: Both results follow easily from theorem 1.2.2, the first because the map $(C^i_{(c)}(X,\underline{\underline{S}}^j)) \to (C^i_{(c)}(X,\underline{\underline{T}}^j))$ gives quasi-isomorphisms between slices for i fixed, and the second because the same map gives quasi-isomorphisms between slices for j fixed.

Remark: All constructions done with n-complexes of vector spaces can be done equally well with n-complexes of sheaves. Hence the hypercohomology of an n-complex of sheaves is a single complex of sheaves.

Although hypercohomology in this sense is only applied to complexes of dimension > 1, if there is a possibility of confusion, this operation will be referred to as algebraic hypercohomology, and the hypercohomology of theorem 1.2.5 will be referred to as topological hypercohomology.

§1.3 Injective and Projective Sheaves

Definition: A sheaf $\underline{\underline{I}}$ is *injective* if, for any map $f : \underline{\underline{A}} \to \underline{\underline{I}}$ and any inclusion $i : \underline{\underline{A}} \hookrightarrow \underline{\underline{B}}$, there is an extension $g : \underline{\underline{B}} \to \underline{\underline{I}}$ such that $f = g \circ i$.

A sheaf $\underline{\underline{P}}$ is *projective* if for any map $f : \underline{\underline{P}} \to \underline{\underline{A}}$ and any surjection $\pi : \underline{\underline{B}} \twoheadrightarrow \underline{\underline{A}}$, there is a map $g : \underline{\underline{P}} \to \underline{\underline{B}}$ such that $f = \pi \circ g$.

Lemma 1.3.1: (i) If $i : \underline{\underline{I}} \hookrightarrow \underline{\underline{A}}$ is an inclusion of an injective sheaf into a sheaf, then $\underline{\underline{I}}$ is a direct summand; i.e., $\underline{\underline{A}} \cong \underline{\underline{I}} \oplus \underline{\underline{\mathrm{Cok}}}(i)$.

(ii) $\bigoplus_{i=1} \underline{\underline{A}}_i$ is injective if and only if each $\underline{\underline{A}}_i$ is injective.

(iii) If $\pi : \underline{\underline{A}} \twoheadrightarrow \underline{\underline{P}}$ is a surjection of a sheaf onto a projective sheaf, then $\underline{\underline{A}} \cong \underline{\underline{P}} \oplus \underline{\underline{\mathrm{Ker}}}(\pi)$.

(iv) $\bigoplus_{i=1} \underline{\underline{A}}_i$ is projective if and only if each $\underline{\underline{A}}_i$ is projective.

Proof: Each of these follows by an easy application of the definitions above. For example,

(i) Since $\underline{\underline{I}}$ is injective, there is a map $g : \underline{\underline{A}} \to \underline{\underline{I}}$ such that $g \circ i = \text{identity}$. Then the map $\underline{\underline{A}} \xrightarrow{g \oplus \pi} \underline{\underline{I}} \oplus \underline{\underline{Cok}}(i)$ is an isomorphism since we have a split exact sequence $0 \to \underline{\underline{I}}(\sigma) \underset{g}{\overset{i}{\rightleftarrows}} \underline{\underline{A}}(\sigma) \overset{\pi}{\to} \underline{\underline{Cok}}(i)(\sigma) \to 0$ over each cell.

One of the advantages of working with cellular sheaves is that injective and projective (cellular) sheaves are particularly simple and plentiful.

Definition: Let $\sigma \in X$ and let V be a vector space. Then the sheaf $[\sigma]^V$ is given by $[\sigma]^V(\tau) = \begin{cases} V & \tau \leq \sigma \\ 0 & \tau \nleq \sigma \end{cases}$, where corestriction maps between copies of V are all identity maps. We will also write $[\sigma] = [\sigma]^{\mathbb{Q}}$.

Theorem 1.3.2: A sheaf $\underline{\underline{I}}$ is injective if and only if it is isomorphic to one of the form $\bigoplus_{\sigma \in X} [\sigma]^{V_\sigma}$.

Proof: Given $[\sigma]^V \xleftarrow{f} \underline{\underline{S}} \xrightarrow{j} \underline{\underline{T}}$, the map $f(\sigma) : \underline{\underline{S}}(\sigma) \to V$ can be extended to $g(\sigma) : \underline{\underline{T}}(\sigma) \to V$. For $\tau < \sigma$, define $g(\tau) : \underline{\underline{T}}(\tau) \to [\sigma]^V(\tau) = V$ by $g(\tau) = g(\sigma) \circ p^{\underline{\underline{T}}}_{\tau,\sigma}$. Then g is an extension of f to $\underline{\underline{T}}$ (where $g(\tau) : \underline{\underline{T}}(\tau) \to [\sigma]^V(\tau) = 0$ is the zero map for $\tau \nleq \sigma$). This shows that $[\sigma]^V$ is injective, and hence $\bigoplus_{\sigma \in X} [\sigma]^{V_\sigma}$ is injective by Lemma 1.3.1 (ii).

To show the converse, assume inductively that every injective sheaf that is 0 on all but $k < n$ cells is isomorphic to $\bigoplus_{\sigma\in X}[\sigma]^{V_\sigma}$, and let $\underline{\underline{I}}$ be injective and 0 on exactly n cells. Let σ be a cell of maximal dimension for which $\underline{\underline{I}}(\sigma) \neq 0$. Let $\underline{\underline{I}}(\sigma) = V$, and $(\sigma)^V$ be the sheaf that has V over σ and 0 over every other cell. Then there is an inclusion $j : (\sigma)^V \hookrightarrow [\sigma]^V$ and a map $\alpha : (\sigma)^V \to \underline{\underline{I}}$ where $\alpha(\sigma)$ is the identity, since σ is of maximal dimension. Extend α to $\tilde{\alpha} : [\sigma]^V \to \underline{\underline{I}}$. On $\tau < \sigma$, the map $p^{\underline{\underline{I}}}_{\tau,\sigma} \circ \tilde{\alpha}(\tau) = \tilde{\sigma}(\alpha) \circ p^{[\sigma]^V}_{\tau,\sigma} = p^{[\sigma]^V}_{\tau,\sigma}$ = identity, so $\tilde{\alpha}(\tau)$ is one-to-one. Then $\tilde{\alpha}$ is one-to-one, so by lemma 1.3.1(i), $\underline{\underline{I}} = [\sigma]^V \oplus \underline{\underline{\mathrm{Cok}}}(\alpha)$. $\underline{\underline{\mathrm{Cok}}}(\alpha)$ is 0 on σ as well as every cell that $\underline{\underline{I}}$ is 0 on, so by induction, $\underline{\underline{\mathrm{Cok}}}(\alpha) = \bigoplus_{\gamma\in X}[\gamma]^{V_\gamma}$.

Since it's vacuously true that a sheaf having 0 on all cells is isomorphic to an (empty) direct sum of $[\sigma]^{V's}$, the induction is complete.

The sheaves $[\sigma]^V$ will be called <u>elementary injectives</u>.

There is a natural way to include a sheaf $\underline{\underline{S}}$ into an injective sheaf. Let $\underline{\underline{I}}_S = \bigoplus_{\sigma\in X}[\sigma]^{\underline{\underline{S}}(\sigma)}$. To define the map $i^S : \underline{\underline{S}} \hookrightarrow \underline{\underline{I}}_S$, we let the component map $i^S_\sigma : \underline{\underline{S}} \to [\sigma]^{\underline{\underline{S}}(\sigma)}$ be given by having $i^S_\sigma(\tau) : \underline{\underline{S}}(\tau) \to \underline{\underline{S}}(\sigma)$ be $d^{\underline{\underline{S}}}_{\tau,\sigma}$ for $\tau \leq \sigma$. This is easily checked to be a map of sheaves, and it is also easily seen that i^S is one-to-one on each cell.

Given a map $h : \underline{\underline{S}} \to \underline{\underline{T}}$, we can define a map $h_I : \underline{\underline{I}}_S \to \underline{\underline{I}}_T$, again by defining it on the components of $\underline{\underline{I}}_S$ and $\underline{\underline{I}}_T$: for $\sigma \neq \tau$,

$[\sigma]^{\underline{\underline{S}}(\sigma)} \to [\sigma]^{\underline{\underline{T}}(\tau)}$ is the 0-map, and for $\sigma = \tau$, $[\sigma]^{\underline{\underline{S}}(\sigma)} \to [\sigma]^{\underline{\underline{T}}(\sigma)}$ is $h(\sigma)$ over each face of σ. It can be verified that this defines a sheaf map, and that the following diagram commutes:

$$\begin{array}{ccc} \underline{\underline{S}} & \xrightarrow{h} & \underline{\underline{T}} \\ \Big\downarrow i^S & & \Big\downarrow i^T \\ \underline{\underline{I}}_S & \xrightarrow{h_I} & \underline{\underline{I}}_T \end{array} .$$

From this we get,

Lemma 1.3.3: Any complex of sheaves $\underline{\underline{S}}^{\bullet}$ can be embedded in a complex of injective sheaves, $\underline{\underline{S}}^{\bullet} \hookrightarrow \underline{\underline{I}}^{\bullet}$.

Proof: Use the natural embeddings of the sheaves into injectives described above:

$$\begin{array}{ccccccccccc} \cdots \to & \underline{\underline{S}}^0 & \xrightarrow{\delta^0} & \underline{\underline{S}}^1 & \xrightarrow{\delta^1} & \underline{\underline{S}}^2 & \longrightarrow & \underline{\underline{S}}^3 & \to \cdots \\ & \Big\downarrow i^{S^0} & & \Big\downarrow i^{S^1} & & \Big\downarrow i^{S^2} & & \Big\downarrow i^{S^3} & \\ \cdots \to & \underline{\underline{I}}_{S^0} & \xrightarrow[\delta_I^0]{} & \underline{\underline{I}}_{S^1} & \xrightarrow[\delta_I^1]{} & \underline{\underline{I}}_{S^2} & \xrightarrow[\delta_I^2]{} & \underline{\underline{I}}_{S^3} & \to \cdots \end{array}$$

It is easily verified that the lower horizontal sequence is a chain complex.

<u>Theorem 1.3.4</u>: For any bounded complex of sheaves $\underline{\underline{S}}^\bullet$, there exists a quasi-isomorphism to a bounded complex of injective sheaves, $\underline{\underline{S}}^\bullet \xrightarrow{q.i.} \underline{\underline{I}}^\bullet$.

This is called an injective resolution of $\underline{\underline{S}}^\bullet$.

<u>Proof</u>: We wish to construct a commutative diagram of sheaves

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\underline{\underline{S}}^0 \arrow[r, hook] \arrow[d] & \underline{\underline{I}}^{00} \arrow[r] \arrow[d] & \underline{\underline{I}}^{01} \arrow[r] \arrow[d] & \underline{\underline{I}}^{02} \arrow[r] \arrow[d] & \cdots \\
\underline{\underline{S}}^1 \arrow[r, hook] \arrow[d] & \underline{\underline{I}}^{10} \arrow[r] \arrow[d] & \underline{\underline{I}}^{11} \arrow[r] \arrow[d] & \underline{\underline{I}}^{12} \arrow[r] \arrow[d] & \cdots \\
\underline{\underline{S}}^2 \arrow[r, hook] \arrow[d] & \underline{\underline{I}}^{20} \arrow[r] \arrow[d] & \underline{\underline{I}}^{21} \arrow[r] \arrow[d] & \underline{\underline{I}}^{22} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where each $\underline{\underline{I}}^{ij}$ is injective, the vertical and horizontal sequences are chain complexes, and the horizontal complexes are exact. The first column $\underline{\underline{I}}^{\bullet 0}$ can be constructed by lemma 1.3.3. Suppose the diagram is constructed for all $\underline{\underline{I}}^{ij}$'s where $j \leq n$. Then each $\underline{\underline{I}}^{in}$ maps onto $\underline{\underline{C}}^{in} = \underline{\underline{\mathrm{Cok}}}(\underline{\underline{I}}^{i,n-1} \to \underline{\underline{I}}^{i,n})$ by a map π^{in} , and by the commutativity of the diagram, there are maps $\partial^{i,n}\colon \underline{\underline{C}}^{i,n} \to \underline{\underline{C}}^{i+1,n}$ which commute with the rest of the diagram. Then $(\underline{\underline{C}}^{\bullet n}, \partial^{\bullet n})$ is a chain complex, and each horizontal sequence $\underline{\underline{S}}^i \hookrightarrow \underline{\underline{I}}^{i,0} \to \cdots \to \underline{\underline{I}}^{i,n} \to \underline{\underline{C}}^{i,n}$ is exact.

Let $s^{\bullet n} : \underline{\underline{C}}^{\bullet,n} \hookrightarrow \underline{\underline{I}}^{\bullet,n+1}$ be the canonical embedding of $\underline{\underline{C}}^{\bullet n}$ into an injective complex. Then this constructs the diagram for all $\underline{\underline{I}}^{ij}$'s such that $j \leq n+1$, where the maps $\underline{\underline{I}}^{i,n} \to \underline{\underline{I}}^{i,n+1}$ are $s^{i,n} \circ \pi^{i,n}$, and by induction, this constructs the complete diagram.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots \arrow[r] & \underline{\underline{I}}^{0,n-1} \arrow[r] \arrow[d, "d^{0,n-1}"'] & \underline{\underline{I}}^{0,n} \arrow[r, "\pi^{0,n}"] \arrow[d, "d^{0,n}"'] \arrow[rr, dashed, bend right, "\partial^{0,n}"'] & \underline{\underline{C}}^{0,n} \arrow[r, hook, "s^{0,n}"] \arrow[d] & \underline{\underline{I}}^{0,n+1} \arrow[d, "d^{0,n+1}"'] \\
\cdots \arrow[r] & \underline{\underline{I}}^{1,n-1} \arrow[r] \arrow[d, "d^{1,n-1}"'] & \underline{\underline{I}}^{1,n} \arrow[r, "\pi^{1,n}"] \arrow[d, "d^{1,n}"'] \arrow[rr, dashed, bend right, "\partial^{1,n}"'] & \underline{\underline{C}}^{1,n} \arrow[r, hook, "s^{1,n}"] \arrow[d] & \underline{\underline{I}}^{1,n+1} \arrow[d, "d^{1,n+1}"'] \\
 & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

Notice that if $\underline{\underline{S}}$ is a sheaf for which $\underline{\underline{S}}(\sigma) = 0$ whenever $\dim(\sigma) \geq k$, then $\underline{\underline{S}}(\sigma) = \underline{\underline{I}}_S(\sigma)$ for $\dim(\sigma) \geq k-1$, and hence $\underline{\underline{\mathrm{Cok}}}(i^S) = 0$ for $\dim(\sigma) \geq k-1$. Therefore, in the diagram just constructed, if the cell complex X is n-dimensional, $\underline{\underline{I}}^{ij} = 0$ for $j > n$. Since $\underline{\underline{S}}^{\bullet}$ is bounded, the diagram is bounded.

The associated single complex $\underline{\underline{I}}^{\bullet}$ of $\underline{\underline{I}}^{\bullet\bullet}$ can now be formed, and there is a natural map $\underline{\underline{S}}^{\bullet} \to \underline{\underline{I}}^{\bullet}$ which, by Corollary 1.2.4, is a quasi-isomorphism over each cell, hence is a quasi-isomorphism of complexes of sheaves.

Theorem 1.3.5: Let $[\tau]^V$ and $[\tau]^W$ be elementary injectives. Then

(i) if $\tau \nleq \sigma$, the only map $[\sigma]^V \to [\tau]^W$ is the 0-map;

(ii) if $\tau \leq \sigma$, there is one map $[\sigma]^V \to [\tau]^W$ for each element of $\mathrm{Hom}(V,W)$, and no others. Given $h : V \to W$, we form a map $f : [\sigma]^V \to [\tau]^W$ by letting $f(\gamma) = h$ for $\gamma \leq \tau$ and 0 for $\gamma \nleq \tau$.

The proof is straightforward. If a map $[\sigma]^V \to [\tau]^W$ is described as being "given by $h : V \to W$" where $\tau \leq \sigma$, it will be understood to mean that the map is the one described above.

Each of the theorems 1.3.2-1.3.5 has a corresponding theorem about projective sheaves. The proofs are similar, and will be omitted.

Definition: Given a vector space V and $\sigma \in X$, the sheaf $\{\sigma\}^V$ is given by $\{\sigma\}^V(\tau) = V$ for $\sigma \leq \tau$ and 0 for $\sigma \nleq \tau$, the corestriction maps between copies of V being identity maps. We will write $\{\sigma\} = \{\sigma\}^{\mathbb{Q}}$.

Theorem 1.3.6: A sheaf is projective if and only if it is isomorphic to one of the form $\bigoplus_{\sigma\in X}\{\sigma\}^{V_\sigma}$.

$\{\sigma\}^V$ will be called an elementary projective sheaf.

Lemma 1.3.7: Any complex of sheaves $\underline{\underline{S}}^\bullet$ is the image of a complex of projective sheaves, $\underline{\underline{P}}^\bullet \twoheadrightarrow \underline{\underline{S}}^\bullet$.

Theorem 1.3.8: For any bounded complex of sheaves $\underline{\underline{S}}^\bullet$ there exists a quasi-isomorphism from a bounded complex of projective sheaves, $\underline{\underline{P}}^\bullet \xrightarrow{q.i.} \underline{\underline{S}}^\bullet$.

This is called a projective resolution of $\underline{\underline{S}}^\bullet$.

Theorem 1.3.9: Let $\{\sigma\}^V$ and $\{\tau\}^W$ be elementary projective sheaves. Then,

(i) if $\tau \nleq \sigma$, the only map $\{\sigma\}^V \to \{\tau\}^W$ is the 0-map;

(ii) if $\tau \leq \sigma$, there is one map $\{\sigma\}^V \to \{\tau\}^W$ for each element of $\mathrm{Hom}(V,W)$ and no others. Given $h : V \to W$, we form $f : \{\sigma\}^V \to \{\tau\}^W$ by letting $f(\gamma) = h$ for $\sigma \leq \gamma$ and 0 for $\sigma \nleq \gamma$.

Theorem 1.3.10: For $\underline{\underline{I}}$ an injective sheaf, $H^p(X,\underline{\underline{I}}) = H^p_c(X,\underline{\underline{I}}) = 0$ for $p > 0$.

Proof: We can assume $\underline{\underline{I}} = [\sigma]^V$, an elementary injective, since $H^p_{(c)}(X,\oplus_i \underline{\underline{I}}^i) = \oplus_i H^p_{(c)}(X,\underline{\underline{I}}^i)$.

$C^p(X,[\sigma]^V)$ associates an element of V to each p-dimensional face τ of σ where $\overline{\tau}$ is compact. These are just the chain groups used to compute the CW cohomology of $Y = |\{\tau \leq \sigma | \overline{\tau}$ is compact$\}|$, and δ can be seen to agree with the CW coboundary maps, so $H^p(X,[\sigma]^V) = H^p_{CW}(Y,V)$. But since X is a compact regular CW complex possibly minus a 0-cell, Y is either a closed cell (if $\overline{\sigma}$ is compact) or a closed cell minus the star of a vertex (if $\overline{\sigma}$ is not compact), which is a deformation retract of a closed cell minus a point on its boundary. Then $H^p_{CW}(Y,V) = 0$ for $p > 0$.

Let $\overline{\overline{\sigma}}$ be the closure of σ in the one-point compactification $\overline{X}$

of X, and $\overline{\sigma}$ the closure of σ in X. Notice that $C^p_c(X,[\sigma]^V) = C^p_{CW}(|\overline{\overline{\sigma}}|,V)$ for $p > 0$, and the boundary maps $C^p \to C^{p+1}$ agree for $p > 0$. Then $H^p_c(X,[\sigma]^V) = H^p_{CW}(|\overline{\overline{\sigma}}|,V) = 0$ for $p > 1$. For $p = 1$, we have to show that everything in the image of $\delta^0_{CW} : C^0_{CW}(|\overline{\overline{\sigma}}|,V) \to C^1_{CW}(|\overline{\overline{\sigma}}|,V)$ is in the image of $\delta^0_c : C^0_c(X,[\sigma]^V) \to C^1_c(X,[\sigma]^V)$. If $\overline{\sigma}$ is compact, these are the same maps. Assume $\overline{\sigma}$ is not compact. If $f \in \operatorname{Im}(\delta^0_{CW})$, say $\delta^0_{CW}(g) = f$, let $\tilde{g} \in C^0_c(X,[\sigma]^V)$ be given by $\tilde{g}(w) = g(w) - g(\text{point at } \infty)$. Then $\delta_c\tilde{g} = f$, and hence $H^1_c(X,[\sigma]^V) = H^1_c(|\overline{\overline{\sigma}}|,V) = 0$.

§1.4 Global Sections

Definition: Let $\underline{\underline{S}}$ be a sheaf on X, and let $U = \{\sigma_i\}$ be any subset of cells of X (U is not necessarily a cell complex). Then a global section α of $\underline{\underline{S}}$ on U is a choice $\alpha(\sigma_i)$ of an element of $\underline{\underline{S}}(\sigma_i)$ for each $\sigma_i \in U$ such that if $\sigma_i < \sigma_j$, $\alpha(\sigma_j) = \rho_{\sigma_i,\sigma_j}(\alpha(\sigma_i))$. A global section with compact support is a global section α such that $\alpha(\sigma_i) = 0$ if the closure of $|\sigma_i|$ in $|U|$ is not compact. The group of global sections is denoted $\Gamma(U,\underline{\underline{S}})$, and the group of global sections with compact support is denoted $\Gamma_c(U,\underline{\underline{S}})$.

If $U = X$, there is a nicer description of $\Gamma(X,\underline{\underline{S}})$ and $\Gamma_c(X,\underline{\underline{S}})$. Since every cell has at least one vertex (in the 1-point compactification each cell has at least two vertices), an element $\alpha \in \Gamma(X,\underline{\underline{S}})$ is completely determined by its values $\{\alpha(v) | v \text{ a vertex}\}$. Then we can consider $\Gamma(X,\underline{\underline{S}})$ to be a subset of $C^0(X,\underline{\underline{S}})$, and $\Gamma_c(X,\underline{\underline{S}})$ a subset of $C_c^0(X,\underline{\underline{S}})$. Under this interpretation, $\alpha \in C^0(X,\underline{\underline{S}})$ is in $\Gamma(X,\underline{\underline{S}})$ if and only if α extends to a well-defined global section $\tilde{\alpha}$ in the original sense, by $\tilde{\alpha}(\sigma) = p_{v,\sigma}\alpha(v)$ for any vertex $v \leq \sigma$. Similarly, $\alpha \in C_c^0(X,\underline{\underline{S}})$ is in $\Gamma_c(X,\underline{\underline{S}})$ if and only if α extends to a well defined global section with compact support $\tilde{\alpha}$ by $\tilde{\alpha}(\sigma) = p_{v,\sigma}\alpha(v)$ for any vertex $v \leq \sigma$.

Theorem 1.4.1: $\Gamma(X,\underline{\underline{S}}) = \mathrm{Ker}(C^0(X,\underline{\underline{S}}) \xrightarrow{\delta^0} C^1(X,\underline{\underline{S}}))$, and $\Gamma_c(X,\underline{\underline{S}}) = \mathrm{Ker}(C_c^0(X,\underline{\underline{S}}) \xrightarrow{\delta_c^0} C_c^1(X,\underline{\underline{S}}))$.

Proof: If $\alpha \in \Gamma(X,\underline{\underline{S}})$ and σ is a 1-cell with compact closure, then σ has two vertices v_1 and v_2 . $\delta^0(\alpha)(\sigma) = [\sigma:v_1]p_{v_1,\sigma}\alpha(v_1) + [\sigma:v_2]p_{v_2,\sigma}\alpha(v_2) = \pm(p_{v_1,\sigma}\alpha(v_1)-p_{v_2,\sigma}\alpha(v_2)) = \pm(\tilde{\alpha}(\sigma)-\tilde{\alpha}(\sigma)) = 0$ ($\tilde{\alpha}$ is the extension of α to all cells), so $\Gamma(X,\underline{\underline{S}}) \subseteq \mathrm{Ker}(\delta^0)$.

To show that $\mathrm{Ker}(\delta^0) \subseteq \Gamma(X,\underline{\underline{S}})$, we need to check that if $\alpha \in \mathrm{Ker}(\delta^0)$, v_1,v_2 are vertices and $v_i < \sigma$, $i = 1,2$ where σ is any cell in X , then $p_{v_1,\sigma}(\alpha(v_1)) = p_{v_2,\sigma}(\alpha(v_2))$. We can always find a path along 1-cells in $\overline{\sigma}$ from v_1 to v_2 (if $|\overline{\sigma}|$ is not compact, consider the 1-point compactification of $|\overline{\sigma}|$; then we need only find a path of 1-cells from v_1 to v_2 avoiding the point at infinity). If v_1,v_2 are connected by a single 1-cell, the result follows immediately. Suppose we've shown the result for v_1,v_2 connected by a path of less than n 1-cells. Let v_1,v_2 be vertices of σ connected by a path of n 1-cells, and let w be a vertex of σ so that v_1,w

are connected by a path of n-1 1-cells, and w and v_2 are the two vertices of a 1-cell τ. Then $p_{v_1,\sigma}(\alpha(v_1)) = p_{w,\sigma}(\alpha(w)) = p_{\tau,\sigma}p_{w,\tau}(\alpha(w)) = p_{\tau,\sigma}p_{v_2,\tau}(\alpha(v_2)) = p_{v_2,\sigma}(\alpha(v_2))$. Then $\mathrm{Ker}(\delta^0) \subseteq \Gamma(X,\underline{\underline{S}})$, and hence, $\mathrm{Ker}(\delta^0) = \Gamma(X,\underline{\underline{S}})$.

As before, if $|\overline{\sigma}|$ is a compact 1-cell, then for $\alpha \in \Gamma_c(X,\underline{\underline{S}})$, $(\delta_c^0\alpha)(\sigma) = 0$. If $|\overline{\sigma}|$ is not compact, then it has only one vertex v, and $(\delta_c^0\alpha)(\sigma) = \pm p_{v,\sigma}\alpha(v) = \pm\tilde{\alpha}(\sigma) = 0$, since $\alpha \in \Gamma_c(X,\underline{\underline{S}})$. Then $\alpha \in \mathrm{Ker}(\delta_c^0)$.

Conversely, given $\alpha \in \mathrm{Ker}(\delta_c^0)$, α determines a well-defined global section $\tilde{\alpha}$ over all cells by $\tilde{\alpha}(\sigma) = p_{v,\sigma}(\alpha(v))$, $v \leq \sigma$, since $\Gamma(X,\underline{\underline{S}}) = \mathrm{Ker}(\delta^0)$ and $\mathrm{Ker}(\delta_c^0) \subseteq \mathrm{Ker}(\delta^0) \subseteq C^0(X,\underline{\underline{S}}) = C_c^0(X,\underline{\underline{S}})$. For any non-compact 1-cell $\overline{\sigma}$ we have $\alpha(\sigma) = 0$. Any non-compact cell $\overline{\tau}$ has a non-compact 1-cell $\overline{\sigma}$ as a face (think of the 1-point compactification of $\overline{\tau}$), so $\tilde{\alpha}(\tau) = p_{\sigma,\tau}\tilde{\alpha}(\sigma) = p_{\sigma,\tau}(0) = 0$. Then $\tilde{\alpha}$ is 0 on non-compact cells $\overline{\tau}$, so $\alpha \in \Gamma_c(X,\underline{\underline{S}})$ and hence $\Gamma_c(X,\underline{\underline{S}}) = \mathrm{Ker}(\delta_c^0)$.

<u>Theorem 1.4.2</u>: Let $\underline{\underline{S}}^\bullet$ be a bounded complex of sheaves and $h^\bullet : \underline{\underline{S}}^\bullet \to \underline{\underline{I}}^\bullet$ a bounded injective resolution. Then we have the following isomorphisms:

$$\mathbb{H}^\bullet(X,\underline{\underline{S}}^\bullet) \cong H^\bullet(\Gamma(X,\underline{\underline{I}}^\bullet)) \quad \text{and}$$
$$\mathbb{H}_c^\bullet(X,\underline{\underline{S}}^\bullet) \cong H^\bullet(\Gamma_c(X,\underline{\underline{I}}^\bullet)) .$$

<u>Proof</u>: We have $\mathbb{H}_{(c)}^\bullet(X,\underline{\underline{S}}^\bullet) \xrightarrow[\cong]{h^\bullet} \mathbb{H}_{(c)}^\bullet(X,\underline{\underline{I}}^\bullet)$ by theorem 1.2.5(i).

The following diagram of vector spaces has exact horizontal complexes by theorems 1.3.10 and 1.4.1:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\Gamma_{(c)}(X,\underline{\underline{I}}^0) \arrow[r, hook, "i^0"] \arrow[d] & C^0_{(c)}(X,\underline{\underline{I}}^0) \arrow[r] \arrow[d] & C^1_{(c)}(X,\underline{\underline{I}}^0) \arrow[r] \arrow[d] & \cdots \\
\Gamma_{(c)}(X,\underline{\underline{I}}^1) \arrow[r, hook, "i^1"] \arrow[d] & C^0_{(c)}(X,\underline{\underline{I}}^1) \arrow[r] \arrow[d] & C^1_{(c)}(X,\underline{\underline{I}}^1) \arrow[r] \arrow[d] & \cdots \\
\Gamma_{(c)}(X,\underline{\underline{I}}^2) \arrow[r, hook, "i^2"] \arrow[d] & C^0_{(c)}(X,\underline{\underline{I}}^2) \arrow[r] \arrow[d] & C^1_{(c)}(X,\underline{\underline{I}}^2) \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

Then by Corollary 1.2.4, $\underline{i^\bullet} : H^\bullet(\Gamma_{(c)}(X,\underline{\underline{I}}^\bullet)) \to \mathbb{H}^\bullet_{(c)}(X,\underline{\underline{I}}^\bullet)$ is an isomorphism.

§1.5 Subdivisions

Definition: A subdivision of a cell complex X is a cell complex X' with $|X'| = |X|$, and where every cell of X is a union of cells of X' . If $\underline{\underline{S}}$ is a sheaf on X and X' is a subdivision of X , then we can define the subdivision of $\underline{\underline{S}}$, a sheaf on X' , to be $\underline{\underline{S}}'$ given by $\underline{\underline{S}}'(\sigma) = \underline{\underline{S}}(\tilde{\sigma})$ where $\sigma \subseteq \tilde{\sigma}$, and $p^{\underline{\underline{S}}'}_{\sigma,\tau} = p^{\underline{\underline{S}}}_{\tilde{\sigma},\tilde{\tau}}$ where $\sigma \subseteq \tilde{\sigma}$, $\tau \subseteq \tilde{\tau}$. If $\underline{\underline{S}}^\bullet$ is a complex of sheaves on X , then the subdivision of $\underline{\underline{S}}^\bullet$, a complex of sheaves on X' , is defined by subdividing the sheeves $\underline{\underline{S}}^p$ and letting the boundary map $d^p(\sigma) : \underline{\underline{S}}^{p'}(\sigma) \to \underline{\underline{S}}^{p+1'}(\sigma)$ be $d^p(\tilde{\sigma}) : \underline{\underline{S}}^p(\tilde{\sigma}) \to \underline{\underline{S}}^{p+1}(\tilde{\sigma})$ for $\sigma \subseteq \tilde{\sigma} \in X$.

<u>Lemma 1.5.1</u>: Let $\underline{\underline{S}}^\bullet$ be a complex of sheaves over X , X' a subdivision of X , and $\underline{\underline{S}}'^\bullet$ the subdivision of $\underline{\underline{S}}^\bullet$. Then there are natural isomorphisms of chain complexes,

$$\lambda^\bullet : \Gamma(X,\underline{\underline{S}}^\bullet) \to \Gamma(X',\underline{\underline{S}}'^\bullet) \quad \text{and}$$
$$\lambda_c^\bullet : \Gamma_c(X,\underline{\underline{S}}^\bullet) \to \Gamma_c(X',\underline{\underline{S}}'^\bullet) \ .$$

<u>Proof</u>: We define maps $\lambda_{(c)} : \Gamma_{(c)}(X,\underline{\underline{S}}) \to \Gamma_{(c)}(X,\underline{\underline{S}}')$ for $\underline{\underline{S}}$ a single sheaf.

Given a global section α of $\underline{\underline{S}}$, we can associate to it the global section α' of $\underline{\underline{S}}'$ by $\alpha'(\sigma') = \alpha(\sigma)$ where $\sigma' \subseteq \sigma$. If $\sigma' \in X'$ and $\overline{\sigma'}$ is not compact, and $\sigma' \subseteq \sigma$, then $\overline{\sigma}$ is not compact (this just says that if σ' has the point at infinity of the one-point compactification $\overline{X}$ as a vertex, then so does σ). Hence $\alpha \in \Gamma_c(X,\underline{\underline{S}})$ implies $\alpha' \in \Gamma_c(X',\underline{\underline{S}}')$. So we have linear maps $\lambda : \Gamma(X,\underline{\underline{S}}) \to (X',\underline{\underline{S}}')$ and $\lambda_c : \Gamma_c(X,\underline{\underline{S}}) \to \Gamma_c(X',\underline{\underline{S}}')$.

Suppose that $\alpha' \in \Gamma(X',\underline{\underline{S}}')$, and $\sigma \in X$, $\sigma = \cup\sigma_i'$, $\sigma_i' \in X'$. Since $\underline{\underline{S}}'(\sigma_i') = \underline{\underline{S}}(\sigma)\ \forall\ i$ and all the corestriction maps between the $\underline{\underline{S}}'(\sigma_i')$'s are the identity, α' associates a unique element of $\underline{\underline{S}}(\sigma)$ to the cell σ . If $\tau < \sigma$, $\tau,\sigma \in X$, then we can find $\tau' \subseteq \tau$, $\sigma' \subseteq \sigma$, $\tau' < \sigma'$ where $\tau',\sigma' \in X'$. α' associates $\alpha'(\tau')$ to τ and $\alpha'(\sigma')$ to σ , and $\alpha'(\sigma') = p_{\tau',\sigma'}\alpha'(\tau') = p_{\tau,\sigma}\alpha'(\tau')$, so every global section of $\underline{\underline{S}}'$ gives rise to a global section of $\underline{\underline{S}}$. This defines a map $\Gamma(X',\underline{\underline{S}}') \to \Gamma(X,\underline{\underline{S}})$ which is clearly the inverse of λ, so λ is an isomorphism.

If $\alpha' \in \Gamma_c(X'\underline{\underline{S}}')$ and $\lambda(\alpha) = \alpha'$ $(\alpha \in \Gamma(X,\underline{\underline{S}}))$, then α is zero on all σ where $\overline{\sigma}$ is not compact since every such σ contains a cell $\sigma' \in X'$ where $\overline{\sigma'}$ is not compact. Then $\alpha \in \Gamma_c(X,\underline{\underline{S}})$, and hence λ_c is an isomorphism.

Given a chain complex $\underline{\underline{S}}^\bullet$, for each $p \in \mathbb{Z}$, define $\lambda^p_{(c)} : \Gamma_{(c)}(X,\underline{\underline{S}}^p) \to \Gamma_{(c)}(X',\underline{\underline{S}}'^p)$ as above. It is easily verified that $\lambda^\bullet$ and $\lambda^\bullet_c$ are chain maps, hence they are isomorphisms of chain complexes.

<u>Theorem 1.5.2</u>: Let $\underline{\underline{S}}^\bullet$ be a bounded complex of sheaves on X, X' a subdivision of X and $\underline{\underline{S}}'^\bullet$ the subdivision of $\underline{\underline{S}}^\bullet$. Then we have the following isomorphisms:

$$\mathbb{H}^\bullet(X,\underline{\underline{S}}^\bullet) \cong \mathbb{H}^\bullet(X',\underline{\underline{S}}'^\bullet), \quad \text{and}$$
$$\mathbb{H}^\bullet_c(X,\underline{\underline{S}}^\bullet) \cong \mathbb{H}^\bullet_c(X',\underline{\underline{S}}'^\bullet).$$

<u>Proof:</u> Let $\underline{\underline{S}}^\bullet \to \underline{\underline{I}}^\bullet$ be a bounded injective resolution, and $\underline{\underline{I}}'^\bullet$ the subdivision of $\underline{\underline{I}}^\bullet$ ($\underline{\underline{I}}'^\bullet$ is <u>not</u>, in general, an injective complex). Then there is a natural map $\underline{\underline{S}}'^\bullet \to \underline{\underline{I}}'^\bullet$ which is a quasi-isomorphism, so we have by theorem 1.2.5(i),

$$\mathbb{H}^{\bullet}_{(c)}(X',\underline{\underline{S}}'^{\bullet}) \to \mathbb{H}^{\bullet}_{(c)}(X',\underline{\underline{I}}'^{\bullet}) .$$

If $\underline{\underline{I}}'$ is a subdivided injective, then it is clearly a direct sum of subdivided elementary injectives. Using exactly the same reasoning as in the proof of theorem 1.3.10, then, it follows that $H^p(X',\underline{\underline{I}}') = H^p_c(X',\underline{\underline{I}}') = 0$ for $p > 0$. Then by theorem 1.4.1 and Corollary 1.2.4, $H^{\bullet}(\Gamma_{(c)}(X',\underline{\underline{I}}'^{\bullet}) \overset{\cong}{\to} H^{\bullet}_{(c)}(X',\underline{\underline{I}}'^{\bullet})$. Finally, by theorem 1.4.2 and lemma 1.5.1 we have, $\mathbb{H}^{\bullet}_{(c)}(X,\underline{\underline{S}}) \cong H^{\bullet}(\Gamma_{(c)}(X,\underline{\underline{I}}^{\bullet})) \overset{\cong}{\to} H^{\bullet}(\Gamma_{(c)}(X',\underline{\underline{I}}'^{\bullet}))$.

§1.6 Cellular Maps

Definition: Let X and Y be cell complexes. A function $f : X \to Y$ along with a continuous map $|f| : |X| \to |Y|$ is a cellular map if

(i) for $\sigma \in X$, $|f|(|\sigma|)$ is the cell $|f(\sigma)|$;

(ii) $|f| \,|_{|\sigma|} : |\sigma| \to |f(\sigma)|$ is the projection $\mathbb{R}^{n+k} \to \mathbb{R}^n$, topologically;

(iii) Given $\sigma \in X$ and $y,z \in |f(\sigma)|$, $|f|^{-1}(y) \cap \overline{|\sigma|}$ is compact if and only if $|f|^{-1}(z) \cap \overline{|\sigma|}$ is.

We will usually write $|f|$ simply as f, when no confusion can result.

Lemma 1.6.1: Let $f : X \to Y$ be a cellular map where $X = \overline{\sigma}$, $Y = \overline{\tau}$, and $f(\sigma) = \tau$. Given $y \in f(|X|)$, let C be a compact connected component of $f^{-1}(y)$, and let U be a neighborhood of C in

$|X|$. Then there exist neighborhoods V of y and $W \subseteq U$ of C such that $f(W) = V$, and $f^{-1}(V) = W$. In particular, if $f^{-1}(y)$ has a compact connected component, then it is connected.

Proof: Fix a metric on $|X|$. Since C is compact and $|X|$ is locally compact, there exists $\varepsilon > 0$ such that $K = \{x \in |X| \,|\, d(x,C) \le \varepsilon\}$ is compact, $K \subseteq U$, and $K \cap f^{-1}(y) = C$. Let $W_0 = \{x \in |X| \,|\, d(x,C) < \varepsilon/2\}$. We will show first that there exists a neighborhood V of y such that $f^{-1}(V) \subseteq W_0$.

Assume to the contrary that for all neighborhoods V of y , $(|X| - W_0) \cap f^{-1}(V) \neq \emptyset$. Suppose V is a neighborhood of y and $V \cap |\tau|$ is connected. We claim that $(K - W_0) \cap f^{-1}(V) \neq \emptyset$. To show this, we can assume that $(|X| - K) \cap f^{-1}(V) \neq \emptyset$ since otherwise the claim follows immediately. Since $(|X| - K) \cap f^{-1}(V)$ is open, $(|X| - K) \cap f^{-1}(V) \cap |\sigma|$ is non-empty. Similarly, $W_0 \cap f^{-1}(V)$ is non-empty (it contains C) and open, so $W_0 \cap f^{-1}(V) \cap |\sigma| \neq \emptyset$. $f : |\sigma| \to |\tau|$ is just the projection $\mathbb{R}^{n+k} \to \mathbb{R}^n$ topologically and $V \cap |\tau|$ is connected, so we have that $|\sigma| \cap f^{-1}(V \cap |\tau|) = |\sigma| \cap f^{-1}(V)$ is connected. Since there are points $x \in |\sigma| \cap f^{-1}(V)$ with $d(x,C) > \varepsilon$ (i.e., $x \in |X| - K$) as well as with $d(x,C) < \varepsilon/2$ (i.e., $x \in W_0$), the connectedness of $|\sigma| \cap f^{-1}(V)$ shows that there are points $x \in |\sigma| \cap f^{-1}(V)$ with $\varepsilon/2 \le d(x,C) \le \varepsilon$, i.e., $(K - W_0) \cap |\sigma| \cap f^{-1}(V) \neq \emptyset$. This shows the claim.

There are arbitrarily small neighborhoods V of y where

$V \cap |\tau|$ is connected, so we can form a sequence $\{x_i\} \subseteq K - W_0$ such that $f(x_i) \to y$. Then $\{x_i\}$ has an accumulation point x in K since K is compact, and $f(x) = y$. But $\varepsilon/2 < d(x_i, C)\ \forall\ i$ so $x \notin C$, contradicting $C = K \cap f^{-1}(y)$.

Hence there is a neighborhood V of y for which $f^{-1}(V) \subseteq W_0$. Letting $W = f^{-1}(V)$, we have $W \subseteq U$ such that $f^{-1}(V) = W$. It remains to show that $f(W) = W$. Let $z \in V$. We can find a sequence $\{z_i\} \subseteq V \cap |\tau|$ such that $\{z_i\} \to z$. Since $z_i \in |\tau|$, there is an $x_i \in |\sigma|$ such that $f(x_i) = z_i$. $\{x_i\} \subseteq W \subseteq K$, so $\{x_i\}$ has an accumulation point x in K , and $f(x) = z$. Then $z \in V$ implies that $x \in f^{-1}(V) = W$, so $z \in f(W)$. This completes the proof.

Let $S(X)$ be the category of cellular sheaves on X , and let $f : X \to Y$ be a cellular map. We can then construct the following factors:

$$f^* : S(Y) \to S(X) .$$

If $\underline{\underline{T}} \in S(Y)$ and $\sigma \in X$, define $(f^*\underline{\underline{T}})(\sigma) = \underline{\underline{T}}(f(\sigma))$. If $\sigma \leq \tau$, then $f(\sigma) \leq f(\tau)$ by continuity, so we can define the corestriction maps as $p^{f^*\underline{\underline{T}}}_{\sigma,\tau} = p^{\underline{\underline{T}}}_{f(\sigma),f(\tau)}$. Given a morphism $\underline{\underline{S}} \to \underline{\underline{T}}$ in $S(Y)$, there is an obvious way of defining the corresponding morphism $f^*\underline{\underline{S}} \to f^*\underline{\underline{T}}$.

Given $\tau \in Y$, we define $St(\tau) = \{\gamma \in Y | \gamma \geq \tau\}$

$$f_* : S(X) \to S(Y) .$$

If $\underline{\underline{T}} \in S(X)$ and $\tau \in Y$ then $(f_*\underline{\underline{T}})(\tau) = \Gamma(f^{-1}(st(\tau)),\underline{\underline{T}})$. The corestriction maps are just restrictions of sections. f_* transforms morphisms in an obvious way.

Thereom 1.6.2: If $\sigma \in X$ and V is a vector space, then $f_*[\sigma]^V = [f(\sigma)]^V$.

Proof: If $\tau \nleq f(\sigma)$, then $\overline{\sigma} \cap f^{-1}(st(\tau)) = \emptyset$, so $\Gamma(f^{-1}(st(\tau)),[\sigma]^V) = 0$. If $\tau \leq f(\sigma)$, then $\sigma \in f^{-1}(st(\tau))$, and each element of V put on σ extends uniquely to a section of $[\sigma]^V$ on $f^{-1}(st(\tau))$, so $f_*[\sigma]^V(\tau) = V$. Corestriction maps between cells $\gamma < \tau \leq f(\sigma)$ are easily seen to be the identity.

$$f_! : S(X) \to S(Y) .$$

If $\underline{\underline{T}} \in S(X)$ and $\tau \in Y$, then $(f_!\underline{\underline{T}})(\tau) = \{h \in \Gamma(f^{-1}(\tau),\underline{\underline{T}}) | h(\sigma) = 0$ if $|\overline{\sigma}| \cap f^{-1}(y)$ is not compact for $y \in |\tau|\}$. This group is well-defined by the third condition in the definition of a cell map. Let $\gamma < \tau$ be cells in Y , and $h \in \Gamma(f^{-1}(\gamma),\underline{\underline{T}})$ be an element of $(f_!\underline{\underline{T}})(\gamma)$. Then $p^{f_!\underline{\underline{T}}}_{\gamma,\tau}(h)$ is the element of $\Gamma(f^{-1}(\tau),\underline{\underline{T}})$ that on $\sigma \in f^{-1}(\tau)$ takes the value $p^{\underline{\underline{T}}}_{\lambda,\sigma}(h(\lambda))$ for any $\lambda < \sigma$ in $f^{-1}(\gamma)$, or, if no such λ exists, takes the value 0 .

We need to show that this is well-defined, and that this does, in fact, form a sheaf.

First, suppose $f(\lambda_1) = f(\lambda_2) = \gamma$, $\lambda_1 < \sigma$, $\lambda_2 < \sigma$ and $f(\sigma) = \tau$. Consider f as restricted to $f : \overline{\sigma} \to \overline{\tau}$. Then by lemma 1.6.1, for $y \in |\gamma|$, $f^{-1}(y)$ is either connected or consists of only non-compact connected components. If $f^{-1}(y)$ is connected, then there is a sequence of faces of σ in $f^{-1}(\gamma)$ from λ_1 to λ_2 : $\lambda_1 \leq \mu_0 \geq \mu_1 \leq \dots \leq \mu_n \geq \lambda_2$. Then $p^{\underline{\underline{T}}}_{\overline{\mu}_i,\sigma}(h(\mu_i)) = p^{\underline{\underline{T}}}_{\overline{\mu}_{i+1},\sigma} p^{\underline{\underline{T}}}_{\overline{\mu}_i,\mu_{i+1}}(h(\mu_i)) = p^{\underline{\underline{T}}}_{\overline{\mu}_{i+1},\sigma}(h(\mu_{i+1}))$ (if $\mu_i \leq \mu_{i+1}$). Applying this successively we get that $p^{\underline{\underline{T}}}_{\overline{\lambda}_1,\sigma}(h(\lambda_1)) = p^{\underline{\underline{T}}}_{\overline{\lambda}_2,\sigma}(h(\lambda_2))$. If $f^{-1}(y)$ is not connected, then for any face $\lambda < \sigma$, $f(\lambda) = \gamma$, let A be the connected component of $f^{-1}(y)$ that intersects $|\lambda|$ (A is unique by condition (ii) of the definition of a cell map), and let $\{\mu_i\}$ be the set of cells that intersect A . By condition (ii) again, $A = f^{-1}(y) \cap \bigcup_i |\mu_i|$. At least one of the μ_i's must have $f^{-1}(y) \cap |\overline{\mu_i}|$ not compact as otherwise $f^{-1}(y) \cap \overline{\bigcup_i |\mu_i|} = f^{-1}(y) \cap \bigcup_i |\overline{\mu_i}| = \bigcup_i (f^{-1}(y) \cap |\overline{\mu_i}|)$ would be compact and since A is a closed subset of this, A would then be compact. Suppose that $\lambda' \in \{\mu_i\}$ has $f^{-1}(y) \cap |\overline{\lambda'}|$ not compact. Since A is connected, we can as before construct a sequence of faces of σ in $f^{-1}(\gamma)$, $\lambda \leq \mu_0 \geq \mu_1 \leq \dots \leq \mu_n \geq \lambda'$, and conclude that $p^{\underline{\underline{T}}}_{\overline{\lambda},\sigma}(h(\lambda)) = p^{\underline{\underline{T}}}_{\overline{\lambda}',\sigma}(h(\lambda')) = 0$. Then in particular, for λ_1, λ_2 as above, we have $p^{\underline{\underline{T}}}_{\overline{\lambda}_1,\sigma}(h(\lambda_1)) = p^{\underline{\underline{T}}}_{\overline{\lambda}_2,\sigma}(h(\lambda_2)) = 0$.

Hence the value $p^{\underline{\underline{T}}}_{\lambda,\sigma}(h(\lambda))$ in the definition is independent of the choice of λ. To see that $p^{f_!\underline{\underline{T}}}_{\gamma,\tau}(h)$ is a section of $f^{-1}(\tau)$, we must show that if $\sigma_1 < \sigma_2$ and $f(\sigma_1) = f(\sigma_2) = \tau$, then the value associated to σ_1 maps to the value associated to σ_2 under the co-restriction map. If there exists $\lambda < \sigma_1$ where $f(\lambda) = \gamma$, then $\lambda < \sigma_2$, and hence $p^{\underline{\underline{T}}}_{\sigma_1,\sigma_2} \circ p^{\underline{\underline{T}}}_{\lambda,\sigma_1}(h(\lambda)) = p^{\underline{\underline{T}}}_{\lambda,\sigma_2}(h(\lambda))$ shows that the values are consistent. If there is no $\lambda < \sigma_1$ such that $f(\lambda) = \gamma$, then the value associated to σ_1 is 0.

Considering f as restricted to $f : \overline{\sigma}_2 \to \overline{\tau}$ and letting $y \in |\gamma|$, if $f^{-1}(y)$ has a connected component that is not compact, then as before, the value associated to σ_2 is 0. If $f^{-1}(y)$ is compact, then it's connected, and $|\overline{\sigma_2}| - |\overline{\sigma_1}|$ is a neighborhood of $f^{-1}(y)$. But this contradicts lemma 1.6.1 since there are points z on τ arbitrarily close to y for which $f^{-1}(z) \not\subseteq |\overline{\sigma_2}| - |\overline{\sigma_1}|$, as $f(\sigma_1) = \tau$. Hence $f^{-1}(y)$ cannot be compact.

We've shown that $p^{f_!\underline{\underline{T}}}_{\gamma,\tau}(h)$ is a section of $f^{-1}(\tau)$. Suppose $f(\sigma) = \tau$, $y \in |\tau|$, and $|\overline{\sigma}| \cap f^{-1}(y)$ is not compact. Choose $z \in |\gamma|$, and as before, consider f restricted to $f : \overline{\sigma} \to \overline{\tau}$. Suppose $f^{-1}(z)$ were compact. Let K be a compact neighborhood of $f^{-1}(z)$ and find a neighborhood V of z such that $f^{-1}(V) \subseteq K$. Then for a $y' \in V \cap |\tau|$, $f^{-1}(y')$ is compact, contradicting the fact that $f^{-1}(y)$ is not compact. Hence $f^{-1}(z)$ is not compact, so $p^{f_!\underline{\underline{T}}}_{\gamma,\tau}(h)$ associates 0 to σ. This shows that $p^{f_!\underline{\underline{T}}}_{\gamma,\tau}$ is in fact a map from $f_!\underline{\underline{T}}(\gamma)$ to $f_!\underline{\underline{T}}(\tau)$.

It remains to show that $p^{f_!\underline{\underline{T}}}_{\mu,\tau} \circ p^{f_!\underline{\underline{T}}}_{\gamma,\mu} = p^{f_!\underline{\underline{T}}}_{\gamma,\tau}$. Let $h \in f_!\underline{\underline{T}}(\gamma)$ and $\sigma \in X$ s.t. $f(\sigma) = \tau$ (we will again consider f as $f : \overline{\sigma} \to \overline{\tau}$). Let $y \in |\gamma|$.

If $f^{-1}(y) = \emptyset$, then $p_{\mu,\tau}p_{\gamma,\mu}(h)$ and $p_{\gamma,\tau}(h)$ are both 0 on σ .

If $f^{-1}(y)$ is not compact, then $p_{\gamma,\tau}(h)$ is 0 on σ . If there is no $\alpha < \sigma$ such that $f(\alpha) = \mu$ then $p_{\mu,\tau}p_{\gamma,\mu}(h)$ is also 0 on σ . If there is an $\alpha < \sigma$ such that $f(\alpha) = \mu$ but no $\lambda < \alpha$ such that $f(\lambda) = \gamma$, then again, $p_{\mu,\tau}p_{\gamma,\mu}(h)$ is 0 on σ . If there are cells $\lambda < \alpha < \sigma$ with $f(\lambda) = \gamma$ and $f(\alpha) = \mu$, then it follows directly that $p_{\mu,\tau}p_{\gamma,\mu}(h)$ and $p_{\gamma,\tau}(h)$ agree on σ .

If $f^{-1}(y)$ is compact and non-empty, we can choose a neighborhood $U \supseteq f^{-1}(y)$, $\overline{U}$ compact, and a neighborhood V of y such that $z \in V = f^{-1}(z) \subseteq U$ and is non-empty. This shows that there is an $\alpha < \sigma$ with $f(\alpha) = \mu$. Let $\{y_i\}$ be a sequence in $|\mu| \cap V$ that approaches y . For each y_i , choose an $x_i \in |\alpha|$ such that $f(x_i) = y_i$. $\{x_i\} \subseteq \overline{U}$, so there is an accumulation point x , and $f(x) = y$. Therefore there is a $\lambda < \alpha$ with $f(\lambda) = \gamma$, and hence $p_{\mu,\tau}p_{\gamma,\mu}(h)$ and $p_{\gamma,\tau}(h)$ agree on σ .

This shows that $f_!\underline{\underline{T}}$ is a sheaf. Given a morphism $\underline{\underline{S}} \to \underline{\underline{T}}$ between sheaves on X , a morphism $f_!\underline{\underline{S}} \to f_!\underline{\underline{T}}$ can be constructed, which for $\tau \in Y$, maps sections of $\underline{\underline{S}}$ on $f^{-1}(\tau)$ to sections of $\underline{\underline{T}}$ on $f^{-1}(\tau)$ in the natural way. It is easily verified that this is a map between sheaves.

CHAPTER TWO

THE DERIVED CATEGORY

§2.1 The Homotopy Category

The bounded homotopy category $K^b(X)$ has as objects all bounded chain complexes of sheaves on X. Given $\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet \in K^b(X)$, the morphisms $\mathrm{Hom}_{K^b(X)}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet)$ consist of all chain maps $\underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet$, modulo all chain maps homotopic to zero. Given a map $f : X \to Y$, each of the functors $f_*, f_!$, Γ, and Γ_c extends naturally to a functor on $K^b(X)$ (the first two mapping into $K^b(Y)$ and the last two into $K^b(VS)$ = the bounded homotopy category of vector spaces), and f^* extends to $f^* : K^b(Y) \to K^b(X)$. If F is one of these five functors, and $\underline{\underline{T}}^\bullet \in K^b(X)$ (or in the case of f^*, $\underline{\underline{T}}^\bullet \in K^b(Y)$), $F\underline{\underline{T}}^\bullet$ will be the chain complex

$$\cdots \to F\underline{\underline{T}}^i \xrightarrow{F\partial^i} F\underline{\underline{T}}^{i+1} \xrightarrow{F\partial^{i+1}} F\underline{\underline{T}}^{i+2} \to \cdots .$$

Given a chain map $\underline{\underline{S}}^\bullet \to \underline{\underline{T}}^\bullet$, F induces a chain map $F\underline{\underline{S}}^\bullet \to F\underline{\underline{T}}^\bullet$ in a natural way, and it's clear that $\underline{\underline{S}}^\bullet \to \underline{\underline{T}}^\bullet$ homotopic to zero implies $F\underline{\underline{S}}^\bullet \to F\underline{\underline{T}}^\bullet$ is homotopic to zero.

Let $\underline{\underline{A}}, \underline{\underline{B}} \in S(X)$. Then $\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}, \underline{\underline{B}})$ is a sheaf on X defined as follows. Given $\sigma \in X$, $\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}, \underline{\underline{B}})(\sigma)$ consists of all possible sets of maps $\{\underline{\underline{A}}(\gamma) \to \underline{\underline{B}}(\gamma) | \gamma \in st(\sigma)\}$ that commute with the corestriction maps, i.e., all "sheaf maps" $\underline{\underline{A}}|_{st(\sigma)} \to \underline{\underline{B}}|_{st(\sigma)}$ if one ignores the fact that $st(\sigma)$ is not necessarily a cell complex. The corestriction maps are simply "restriction of data", i.e., $p_{\sigma,\tau}(\{\underline{\underline{A}}(\gamma) \to \underline{\underline{B}}(\gamma)\})$ will consist of the same maps, but only those between cells of $st(\tau)$. Given a map $\underline{\underline{B}} \to \underline{\underline{C}}$, there is an associated map $\underline{\underline{Hom}}(\underline{\underline{A}},\underline{\underline{B}}) \to \underline{\underline{Hom}}(\underline{\underline{A}},\underline{\underline{C}})$, and given a map $\underline{\underline{A}} \to \underline{\underline{D}}$, there is an associated map $\underline{\underline{Hom}}(\underline{\underline{D}},\underline{\underline{B}}) \to \underline{\underline{Hom}}(\underline{\underline{A}},\underline{\underline{B}})$. We then have two functors, $\underline{\underline{Hom}}(\underline{\underline{A}},\cdot) : S(X) \to S(X)$ for any $\underline{\underline{A}} \in S(X)$, and $\underline{\underline{Hom}}(\cdot,\underline{\underline{B}}) : S(X) \to S(X)$ for any $\underline{\underline{B}} \in S(X)$, the first one covariant and the second one contravariant.

<u>Theorem 2.1.1</u>:

$$\underline{\underline{Hom}}([\sigma]^V,[\tau]^W) = \begin{cases} 0 & \tau \nleq \sigma \\ [\tau]^{Hom(V,W)} & \tau \leq \sigma \end{cases}$$

<u>Proof</u>: If $\gamma \notin \overline{\sigma} \cap \overline{\tau}$, then for every $\mu \in st(\gamma)$, either $[\sigma]^V(\mu)$ or $[\tau]^W(\mu)$ is 0, so $\underline{\underline{Hom}}([\sigma]^V,[\tau]^W)(\gamma) = 0$. If $\gamma \in \overline{\sigma} \cap \overline{\tau}$, then the only possible way of extending a map $h : [\sigma]^V(\gamma) \to [\tau]^W(\gamma)$ $(h \in Hom(V,W))$ to a commutative diagram of maps $[\sigma]^V|_{st(\gamma)} \to [\tau]^W|_{st(\gamma)}$ is to put h over every cell $\mu \in \overline{\sigma} \cap \overline{\tau} \in st(\gamma)$. If $\tau \leq \sigma$, this gives a commutative diagram, but if $\tau \nleq \sigma$, the diagram over the face relation $\gamma < \tau$ is

$$\begin{tikzcd} V \arrow[r] \arrow[d, "h"'] & 0 \arrow[d] \\ W \arrow[r, "id"] & W \end{tikzcd}$$

which does not commute unless $h = 0$. Then $\underline{\underline{\mathrm{Hom}}}([\sigma]^V, [\tau]^W)(\gamma)$ is $\mathrm{Hom}(V,W)$ if $\gamma \le \tau \le \sigma$ and 0 otherwise, so the result follows (the non-trivial corestriction maps are easily seen to be the identity).

Now, given $\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet \in K^b(X)$, we define an element $\underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet) \in K^b(X)$ by forming the double complex which has $\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{-i}, \underline{\underline{B}}^j)$ in the $ij^{\underline{th}}$ position, and taking the associated single complex of it. Given a chain map $\underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet$ we get a map $(\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{-i}, \underline{\underline{B}}^j)) \to (\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{-i}, \underline{\underline{C}}^j))$ between double complexes which induces a chain map $\underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet) \to \underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{A}}^\bullet, \underline{\underline{C}}^\bullet)$. Suppose the map $\underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet$ is homotopic to zero. We claim that the induced map is homotopic to zero.

Lemma 2.1.2: Let $f^{\bullet\bullet} : \underline{\underline{A}}^{\bullet\bullet} \to \underline{\underline{B}}^{\bullet\bullet}$ be a map between double complexes of sheaves (or vector spaces) with boundary maps $d^{ij}_{k,A}$, $d^{ij}_{k,B}$, $k = 1,2$. Let $t^{\bullet j} : \underline{\underline{A}}^{\bullet j} \to \underline{\underline{B}}^{\bullet, j-1}$ be chain maps such that $d^{i,j-1}_{2,B} t^{ij} + t^{i,j+1} d^{ij}_{2,A} = f^{ij}\ \forall\, i,j$. Then the induced map $f^\bullet : \underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet$ on single complexes is homotopic to zero.

Proof: Let μ^{ij}_k be the signs used to form a single complex. The maps $\mu^{i,j-1}_2 t^{ij}$ for $i+j = K$ map a component of $\underline{\underline{A}}^k$ to a component of $\underline{\underline{B}}^{k-1}$, so they form a map $\tilde{t}^k : \underline{\underline{A}}^k \to \underline{\underline{B}}^{k-1}$. Let D^i_A, D^i_B be the boundary maps for $\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet$, respectively. Then on the component $\underline{\underline{A}}^{ij}$ of $\underline{\underline{A}}^k$ $(i+j = k)$, $D_B^{k-1} t^k + t^{k+1} D_A^k : \underline{\underline{A}}^{ij} \to \underline{\underline{B}}^{i+1,j-1} \oplus \underline{\underline{B}}^{ij}$ is

$$(\mu_1^{i,j-1} d_{1B}^{i,j-1} \oplus \mu_2^{i,j-1} d_{2B}^{i,j-1}) \mu_2^{i,j-1} t^{ij} + (\mu_2^{i+1,j-1} t^{i+1,j} \mu_1^{ij} d_{1A}^{ij} \oplus \mu_2^{ij} t^{i,j+1} \mu_2^{ij} d_{2A}^{ij})$$

$$= (\mu_1^{i,j-1} \mu_2^{i,j-1} d_{1B}^{i,j-1} t^{ij} + \mu_2^{i+1,j-1} \mu_1^{ij} t^{i+1,j} d_{1A}^{ij}) \oplus (d_{2B}^{i,j-1} t^{ij} + t^{i,j+1} d_{2A}^{ij})$$

$= 0 \oplus f^{ij}$, since the two signs in the first term are opposite.

To show that $\underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet) \to \underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{A}}^\bullet, \underline{\underline{C}}^\bullet)$ is homotopic to 0 if $\underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet$ is, note that we have maps $\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{-i}, \underline{\underline{B}}^j) \to \underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{-i}, \underline{\underline{C}}^{j-1})$ induced from a homotopy $\underline{\underline{B}}^j \to \underline{\underline{C}}^{j-1}$, and the compositions

$$\begin{CD}
\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{-i}, \underline{\underline{B}}^j) @>>> \underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{-i}, \underline{\underline{C}}^{j-1}) \\
@VVV @VVV \\
\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{-i-1}, \underline{\underline{B}}^j) @>>> \underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{-i-1}, \underline{\underline{C}}^{j-1})
\end{CD}$$

commute, hence these give chain maps $\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^j) \to \underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^\bullet, \underline{\underline{C}}^{j-1})$. For a fixed $\underline{\underline{A}}^{-i}$, it is easily verified that the maps $\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{-i}, \underline{\underline{B}}^j) \to \underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{-i}, \underline{\underline{C}}^{j-1})$ give a homotopy of the maps induced by $\underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet$, to zero. The lemma then gives the result.

Given a map $\underline{\underline{A}}^\bullet \to \underline{\underline{D}}^\bullet$ we can also construct an induced chain map $\underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{D}}^\bullet, \underline{\underline{B}}^\bullet) \to \underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet)$ in a similar manner, and an analogous application of the above lemma shows that this map is homotopic to 0 if $\underline{\underline{A}}^\bullet \to \underline{\underline{D}}^\bullet$ is .

We then have two functors, $\underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{A}}^\bullet,\cdot) : K^b(X) \to K^b(X)$ and $\underline{\underline{\mathrm{Hom}}}^\bullet(\cdot,\underline{\underline{B}}^\bullet) : K^b(X) \to K^b(X)$, the first covariant and the second contravariant.

For $\underline{\underline{A}},\underline{\underline{B}} \in S(X)$ let $\mathrm{Hom}(\underline{\underline{A}},\underline{\underline{B}})$ be the vector space of all sheaf maps $\underline{\underline{A}} \to \underline{\underline{B}}$. Then $\mathrm{Hom}(\underline{\underline{A}},\underline{\underline{B}}) = \Gamma(X,\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}},\underline{\underline{B}}))$, and we have functors $\mathrm{Hom}(\underline{\underline{A}},\cdot)$ and $\mathrm{Hom}(\cdot,\underline{\underline{B}}) : S(X) \to VS$. Just as with $\underline{\underline{\mathrm{Hom}}}^\bullet$, we can form a chain complex of vector spaces $\mathrm{Hom}^\bullet(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)$ for $\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet \in K^b(X)$, and this gives rise to functors $\mathrm{Hom}^\bullet(\underline{\underline{A}}^\bullet,\cdot) : K^b(X) \to K^b(VS)$ and $\mathrm{Hom}^\bullet(\cdot,\underline{\underline{B}}^\bullet) : K^b(X) \to K^b(VS)$.

Theorem 2.1.3: There are isomorphisms $\mathrm{Hom}_{K^b(X)}(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet) \to H^0(\mathrm{Hom}^\bullet(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet))$ which commute with the functors $\mathrm{Hom}_{K^b(X)}(\cdot,\underline{\underline{B}}^\bullet)$, $H^0(\mathrm{Hom}^\bullet(\cdot,\underline{\underline{B}}^\bullet))$, $\mathrm{Hom}_{K^b(X)}(\underline{\underline{A}}^\bullet,\cdot)$, and $H^0(\mathrm{Hom}^\bullet(\underline{\underline{A}}^\bullet,\cdot))$.

Proof: When forming single complexes from double complexes, we will use the sign convention $\mu_1^{ij} = +1 \ \forall\ i,j$ and $\mu_2^{ij} = (-1)^{i+j+1}$. Let $D^\bullet$ be the boundary maps of $\mathrm{Hom}^\bullet(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)$ and let $h^\bullet \in \mathrm{Hom}^0(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)$. Then the component of $D^0h^\bullet$ in $\mathrm{Hom}(\underline{\underline{A}}^i,\underline{\underline{B}}^{i+1})$ is $\mu_2^{-i,i}d^ih^i + \mu_1^{-i-1,i+1}h^{i+1}d^i = -d^ih^i + h^{i+1}d^i$, so $h^\bullet \in \mathrm{Ker}D^0$ if and only if $h^\bullet$ is a chain map. If $t^\bullet \in \mathrm{Hom}^{-1}(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)$, then the component of $D^{-1}t^\bullet$ in $\mathrm{Hom}(\underline{\underline{A}}^i,\underline{\underline{B}}^i)$ is $\mu_2^{-i,i-1}d^{i-1}t^i + \mu_1^{-i-1,i}t^{i+1}d^i$ (where $t^i \in \mathrm{Hom}(\underline{\underline{A}}^i,\underline{\underline{B}}^{i-1})$) $= d^{i-1}t^i + t^{i+1}d^i$. Then $h^\bullet \in \mathrm{Ker}D^0$ is in $\mathrm{Im}(D^{-1})$ if and only if $h^\bullet$ is homotopic to zero. Therefore, the map $\mathrm{Hom}_{K^b(X)}(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet) \to H^0(\mathrm{Hom}^\bullet(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet))$ which takes $[h^\bullet] \in \mathrm{Hom}_{K^b(X)}(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)$ to the class represented by $h^\bullet \in \mathrm{Hom}^0(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)$ is an isomorphism.

If $g^\bullet : \underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet$ is a chain map, then the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{K^b(X)}(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet) \arrow[r] \arrow[d] & H^0(\mathrm{Hom}^\bullet(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)) \arrow[d] \\
\mathrm{Hom}_{K^b(X)}(\underline{\underline{A}}^\bullet,\underline{\underline{C}}^\bullet) \arrow[r] & H^0(\mathrm{Hom}^\bullet(\underline{\underline{A}}^\bullet,\underline{\underline{C}}^\bullet))
\end{tikzcd}$$

commutes, since $[h^\bullet] \in \mathrm{Hom}_{K^b(X)}(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)$ goes to the element of $H^0(\mathrm{Hom}^\bullet(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet))$ represented by $g^\bullet h^\bullet \in \mathrm{Hom}^0(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)$, by either path. The fact that the isomorphisms commute with the contravariant functors follows similarly.

Given $\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet \in K^b(X)$, we define the tensor product $\underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet \in K^b(X)$ to be the associated single complex of the double complex which has $\underline{\underline{A}}^i \otimes \underline{\underline{B}}^j$ as its $ij^{\underline{th}}$ entry. Given a chain map $\underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet$, there is an associated chain map $\underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet \otimes \underline{\underline{A}}^\bullet \otimes \underline{\underline{C}}^\bullet$, and it follows from lemma 2.1.2 that if $\underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet$ is homotopic to zero, then $\underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet \to \underline{\underline{A}}^\bullet \otimes \underline{\underline{C}}^\bullet$ is homotopic to zero. We then have a covariant functor $\underline{\underline{A}}^\bullet \otimes \cdot : K^b(X) \to K^b(X)$. In the same manner, we can form a covariant functor $\cdot \otimes \underline{\underline{B}}^\bullet : K^b(X) \to K^b(X)$.

In lemmas 2.1.4-2.1.7, all maps are in $K^b(X)$. Hence two maps f,g are equal if chain maps representing f and g are homotopic. Note that since homotopic maps induce the same maps on stalk cohomology, the notion of quasi-isomorphism is well-defined in $K^b(X)$.

<u>Lemma 2.1.4</u>: Let $f : \underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet$, $g : \underline{\underline{A}}^\bullet \to \underline{\underline{C}}^\bullet$ be maps in $K^b(X)$, g a quasi-isomorphism. Then there exists a $\underline{\underline{D}}^\bullet \in K^b(X)$ and maps $r : \underline{\underline{C}}^\bullet \to \underline{\underline{D}}^\bullet$, $s : \underline{\underline{B}}^\bullet \to \underline{\underline{D}}^\bullet$, s a quasi-isomorphism, such that $sf = rg$.

$$\begin{tikzcd}
\underline{\underline{A}}^\bullet \arrow[r, "f"] \arrow[d, "g"'] & \underline{\underline{B}}^\bullet \arrow[d, dashed, "q.i"] \\
\underline{\underline{C}}^\bullet \arrow[r, dashed, "r"] & \underline{\underline{D}}^\bullet
\end{tikzcd}$$

The statement also holds if the directions of all the maps are reversed.

<u>Proof</u>: See [Iv], Chapter 1, Proposition 7.1.

<u>Lemma 2.1.5</u>: (i) Let $f : \underline{\underline{A}}^\bullet \to \underline{\underline{I}}^\bullet$ be a map in $K^b(X)$ where $\underline{\underline{H}}^i(\underline{\underline{A}}^\bullet) = 0$ for all i , and $\underline{\underline{I}}^\bullet$ is a complex of injective sheaves. Then $f = 0$.

(ii) If $f : \underline{\underline{P}}^\bullet \to \underline{\underline{A}}^\bullet$ is a map in $K^b(X)$, $\underline{\underline{P}}^\bullet$ a complex of projective sheaves, and $\underline{\underline{H}}^i(\underline{\underline{A}}^\bullet) = 0 \ \forall\, i$, then $f = 0$.

<u>Proof</u>: (i) We need to show that if $f^\bullet : \underline{\underline{A}}^\bullet \to \underline{\underline{I}}^\bullet$ is a chain map with $\underline{\underline{A}}^\bullet$ and $\underline{\underline{I}}^\bullet$ as above, then $f^\bullet$ is homotopic to zero. Suppose sheaf maps $t^i : \underline{\underline{A}}^i \to \underline{\underline{I}}^{i-1}$ have been constructed for $i \le n$ so that $d^{i-1}t^i + t^{i+1}d^i = f^i$ for $i < n$. Then $d^{n-1}t^n d^{n-1} = d^{n-1}(f^{n-1} - d^{n-2}t^{n-1}) = d^{n-1}f^{n-1} = f^n d^{n-1}$, so $f^n - d^{n-1}t^n = 0$ on $\underline{\underline{\mathrm{Im}}}(d^{n-1}) = \underline{\underline{\mathrm{Ker}}}(d^n)$. Then both d^n and $f^n - d^{n-1}t^n$ can be factored through $\underline{\underline{A}}^n/\underline{\underline{\mathrm{Ker}}}(d^n)$ as shown in the diagram.

$$\begin{tikzcd}
\underline{\underline{A}}^n \arrow[rr, "f^n-d^{n-1}t^n"] \arrow[dr, "\pi"] \arrow[dd, "d^n"'] & & \underline{\underline{I}}^n \\
& \underline{\underline{A}}^n/\underline{\underline{\mathrm{Ker}}}(d^n) \arrow[ur, "r"] \arrow[dl, hook', "\tilde{d}^n"'] & \\
\underline{\underline{A}}^{n+1} \arrow[uurr, dashed, bend right, "t^{n+1}"'] & &
\end{tikzcd}$$

$\tilde{d}^n$ is an inclusion, so there is a map t^{n+1} such that $t^{n+1}\tilde{d}^n = r$. Then $t^{n+1}d^n = t^{n+1}\tilde{d}^n\pi = r\pi = f^n - d^{n-1}t^n$, and hence $f^n = t^{n+1}d^n + d^{n-1}t^n$.

Since $\underline{\underline{A}}^\bullet$ and $\underline{\underline{I}}^\bullet$ are bounded, this process constructs a homotopy $f^\bullet \simeq 0$. The proof of (ii) is similar.

<u>Lemma 2.1.6</u>: If $f : \underline{\underline{I}}^\bullet \to \underline{\underline{A}}^\bullet$ is a quasi-isomorphism in $K^b(X)$ with $\underline{\underline{I}}^\bullet$ a complex of injective sheaves, then there is a map $g : \underline{\underline{A}}^\bullet \to \underline{\underline{I}}^\bullet$ which is the inverse of f (in $K^b(X)$!) .

Proof: See [H], Chapter 1, Lemma 4.5.

Lemma 2.1.7: (i) Given maps $\underline{\underline{A}}^{\bullet} \xrightarrow{f} \underline{\underline{B}}^{\bullet} \xrightarrow{g} \underline{\underline{I}}^{\bullet}$ in $K^b(X)$, f a quasi-isomorphism and $\underline{\underline{I}}^{\bullet}$ a complex of injective sheaves, if $gf = 0$, then $g = 0$.

(ii) If $\underline{\underline{P}}^{\bullet} \xrightarrow{g} \underline{\underline{A}}^{\bullet} \xrightarrow{f} \underline{\underline{B}}^{\bullet}$ is in $K^b(X)$, f a quasi-isomorphism, $\underline{\underline{P}}^{\bullet}$ a complex of projective sheaves, and $fg = 0$, then $g = 0$.

Proof: (i) We want to show that if $\underline{\underline{A}}^{\bullet} \xrightarrow{f^{\bullet}} \underline{\underline{B}}^{\bullet} \xrightarrow{g^{\bullet}} \underline{\underline{I}}^{\bullet}$ are chain maps, $f^{\bullet}$ a quasi-isomorphism, and $g^{\bullet}f^{\bullet}$ is homotopic to zero, then $g^{\bullet}$ is homotopic to zero. Let $t^i : \underline{\underline{A}}^i \to \underline{\underline{I}}^{i-1}$ be such that $d^{i-1}t^i + t^{i+1}d^i = g^i f^i$. Form the mapping cone $M(f^{\bullet}) \in K^b(X)$ where $M(f^{\bullet})^i = \underline{\underline{A}}^{i+1} \oplus \underline{\underline{B}}^i$ and boundary maps are

$$\begin{array}{ccc} \underline{\underline{A}}^{i+1} & \oplus & \underline{\underline{B}}^{i} \\ d^{i+1}\downarrow & \searrow^{(-1)^{i+1}f^{i+1}} & \downarrow d^i \\ \underline{\underline{A}}^{i+2} & \oplus & \underline{\underline{B}}^{i+1} \end{array} .$$

There is a short exact sequence $0 \to \underline{\underline{B}}^{\bullet} \xrightarrow{\alpha^{\bullet}} M(f^{\bullet}) \xrightarrow{\beta^{\bullet}} \underline{\underline{A}}^{\bullet}[1] \to 0$ of complexes of sheaves and chain maps and it follows from the long exact sequence associated to this and the fact that $f^{\bullet}$ is a quasi-isomorphism that $\underline{\underline{H}}^i(M(f^\bullet)) = 0 \ \forall\, i$.

We can define maps $h^i : M(f^\bullet)^i \to \underline{\underline{I}}^i$ by

$$\begin{tikzcd} \underline{\underline{A}}^{i+1} \arrow[dr, "(-1)^{i+1}t^{i+1}"'] & \oplus & \underline{\underline{B}}^i \arrow[dl, "g^i"] \\ & \underline{\underline{I}}^i & \end{tikzcd}$$

and it is easily verified that $h^\bullet$ is a chain map. Then by lemma 2.1.5 (i), $h^\bullet$ is homotopic to zero. But $g^\bullet$ can be factored through $M(f^\bullet)$ as $g^\bullet = h^\bullet \alpha^\bullet$, so $g^\bullet$ is homotopic to zero.

The proof of (ii) is similar.

Definition: We will call two objects $\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet \in K^b(X)$ quasi-isomorphic if there is a $\underline{\underline{C}}^\bullet \in K^b(X)$ and quasi-isomorphisms $f : \underline{\underline{C}}^\bullet \to \underline{\underline{A}}^\bullet$ and $g : \underline{\underline{C}}^\bullet \to \underline{\underline{B}}^\bullet$. It follows from lemma 2.1.4 that this is an equivalence relation, and that an equivalent definition is acquired if the direction of the maps f and g are reversed in the statement above. We will denote the relation by $\underline{\underline{A}}^\bullet \ \underline{\underline{qi}}\ \underline{\underline{B}}^\bullet$.

§2.2 The Derived Category

Let $INJ^b(X)$ be the full subcategory of $K^b(X)$ consisting of complexes of injective sheaves, i.e., the subcategory where

$\mathrm{Hom}_{\mathrm{INJ}^b(X)}(\underline{\underline{I}}^\bullet,\underline{\underline{J}}^\bullet) = \mathrm{Hom}_{K^b(X)}(\underline{\underline{I}}^\bullet,\underline{\underline{J}}^\bullet)$. An immediate consequence of lemma 2.1.6 is that the isomorphisms in $\mathrm{INJ}^b(X)$ are exactly the quasi-isomorphisms. Unfortunately, this does not hold for $K^b(X)$ -- not all quasi-isomorphisms are invertible. However, since every element of $K^b(X)$ has an injective resolution, there is a natural way to form a category $\mathscr{C}$ equivalent to $\mathrm{INJ}^b(X)$ whose objects are the same as $K^b(X)$, and where objects are isomorphic if and only if they're quasi-isomorphic. Given $\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet \in \mathscr{C}$, let $\mathrm{Hom}_{\mathscr{C}}(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)$ consist of all diagrams

$$\begin{tikzcd} \underline{\underline{A}}^\bullet \arrow[r, "h"] & \underline{\underline{I}}^\bullet \arrow[d, "f"] \\ \underline{\underline{B}}^\bullet \arrow[r, "k"] & \underline{\underline{J}}^\bullet \end{tikzcd}$$

in $K^b(X)$, $\underline{\underline{I}}^\bullet$ and $\underline{\underline{J}}^\bullet$ injective complexes and h and k quasi-isomorphisms, where two diagrams are identified if one can construct a commutative diagram (in $K^b(X)$) as follows:

$$\begin{tikzcd}
\underline{\underline{A}}^\bullet \arrow[rr, "h'"] \arrow[dr, "h"] & & \tilde{\underline{\underline{I}}}^\bullet \arrow[dd, "f'"] \\
& \underline{\underline{I}}^\bullet \arrow[ur, "g"] \arrow[dd, "f"] & \\
\underline{\underline{B}}^\bullet \arrow[rr, "k'"] \arrow[dr, "k"] & & \tilde{\underline{\underline{J}}}^\bullet \\
& \underline{\underline{J}}^\bullet \arrow[ur, "g'"] &
\end{tikzcd}$$

By lemma 2.1.6, this identification is an equivalence relation.

Given injective resolutions $\underline{\underline{A}}^\bullet \xrightarrow{h} \underline{\underline{I}}^\bullet$ and $\underline{\underline{B}}^\bullet \xrightarrow{k} \underline{\underline{J}}^\bullet$, we claim that the map $\mathrm{Hom}_{\mathrm{INJ}^b(X)}(\underline{\underline{I}}^\bullet, \underline{\underline{J}}^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet)$ which sends f to $\underline{\underline{A}}^\bullet \xrightarrow{h} \underline{\underline{I}}^\bullet \xrightarrow{f} \underline{\underline{J}}^\bullet \xleftarrow{k} \underline{\underline{B}}^\bullet$ is a bijection, and for different injective resolutions $\underline{\underline{A}}^\bullet \to \tilde{\underline{\underline{I}}}^\bullet$, $\underline{\underline{B}}^\bullet \to \tilde{\underline{\underline{J}}}^\bullet$, there are canonical isomorphisms $\mathrm{Hom}_{\mathrm{INJ}^b(X)}(\underline{\underline{I}}^\bullet, \underline{\underline{J}}^\bullet) \longrightarrow \mathrm{Hom}_{\mathrm{INJ}^b(X)}(\tilde{\underline{\underline{I}}}^\bullet, \tilde{\underline{\underline{J}}}^\bullet)$ commuting with the above maps.

To see this, given quasi-isomorphisms $\underline{\underline{A}}^\bullet \xrightarrow{h} \underline{\underline{I}}^\bullet$ and $\underline{\underline{A}}^\bullet \xrightarrow{h'} \tilde{\underline{\underline{I}}}^\bullet$, by lemma 2.1.4 we can construct a diagram

$$\begin{tikzcd}[column sep=small] & A^\bullet \arrow[dl, "h"'] \arrow[dr, "h'"] & \\ I^\bullet \arrow[dr, "r"'] & & \tilde{I}^\bullet \arrow[dl, "s"] \\ & K^\bullet & \end{tikzcd}$$

with s a quasi-isomorphism. Then r would be forced to be a quasi-isomorphism as well. We can further assume that $K^\bullet$ is an injective complex, since if it isn't, we can resolve it into one. Therefore we have a quasi-isomorphism $s^{-1}r : I^\bullet \to \tilde{I}^\bullet$ commuting with h and h'. If $g, \tilde{g} : I^\bullet \to \tilde{I}^\bullet$ both commuted with h and h', then $\tilde{g}^{-1}gh = \tilde{g}^{-1}h' = \tilde{g}^{-1}\tilde{g}h = h$, so $(\tilde{g}^{-1}g - \mathrm{id})h = 0$. Then by lemma 2.1.7 (i), $\tilde{g}^{-1}g = \mathrm{id}$, so $g = \tilde{g}$.

Given injective resolutions $A^\bullet \xrightarrow{h} I^\bullet$, $A^\bullet \xrightarrow{h'} \tilde{I}^\bullet$, $B^\bullet \to J^\bullet$, $B^\bullet \to \tilde{J}^\bullet$, let g, g' be the unique maps such that $gh = h'$ and $gk = k'$. Let $f : I^\bullet \to J^\bullet$ represent an element of $\mathrm{Hom}_{\mathscr{C}}(A^\bullet, B^\bullet)$ via the resolutions h, k. Then there is a unique map, namely $g'fg^{-1}$, representing the same element via the resolutions h', k'. The claim then follows, where the canonical map $\mathrm{Hom}_{\mathrm{INJ}^b(X)}(I^\bullet, J^\bullet) \to \mathrm{Hom}_{\mathrm{INJ}^b(X)}(\tilde{I}^\bullet, \tilde{J}^\bullet)$ for the above resolutions, is $f \longmapsto g'fg^{-1}$.

To compose two maps, $A^\bullet \to B^\bullet \to C^\bullet$ in $\mathscr{C}$, we simply take the same resolution of $B^\bullet$ in each case, and compose in $\mathrm{INJ}^b(X)$. It's clear that this process is well-defined.

In the following, a map in $K^b(X)$ written as $\underline{\underline{A}}^\bullet \xrightarrow{qi} \underline{\underline{B}}^\bullet$ will be taken to be a quasi-isomorphism. If $\underline{\underline{A}}^\bullet$ and $\underline{\underline{B}}^\bullet$ are isomorphic in $\mathscr{C}$, we have a diagram $\underline{\underline{A}}^\bullet \xrightarrow{qi} \underline{\underline{I}}^\bullet \rightarrow \underline{\underline{J}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet$ where the middle map is an isomorphism in $INJ^b(X)$, i.e., it is a quasi-isomorphism, so $\underline{\underline{A}}^\bullet \overset{qi}{=} \underline{\underline{B}}^\bullet$. If, conversely, there are quasi-isomorphisms $\underline{\underline{A}}^\bullet \xrightarrow{qi} \underline{\underline{K}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet$, then by resolving $\underline{\underline{K}}^\bullet$ if necessary, we can assume $\underline{\underline{K}}^\bullet$ to be injective. $\underline{\underline{A}}^\bullet$ and $\underline{\underline{B}}^\bullet$ are then isomorphic in $\mathscr{C}$ since we have the diagram. $\underline{\underline{A}}^\bullet \xrightarrow{qi} \underline{\underline{K}}^\bullet \xrightarrow{id} \underline{\underline{K}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet$. Hence $\underline{\underline{A}}^\bullet$ and $\underline{\underline{B}}^\bullet$ are isomorphic in $\mathscr{C}$ if and only if they are quasi-isomorphic.

To see that $INJ^b(X)$ and $\mathscr{C}$ are equivalent, one can verify that the functors $F : INJ^b(X) \to \mathscr{C}$ by $F(\underline{\underline{I}}^\bullet) = \underline{\underline{I}}^\bullet$, $(f : \underline{\underline{I}}^\bullet \to \underline{\underline{J}}^\bullet) \mapsto (\underline{\underline{I}}^\bullet \xrightarrow{id} \underline{\underline{I}}^\bullet \xrightarrow{f} \underline{\underline{J}}^\bullet \xleftarrow{id} \underline{\underline{J}}^\bullet)$ and $G : \mathscr{C} \to INJ^b(X)$ by $G(\underline{\underline{A}}^\bullet)$ = the standard injective resolution $\underline{\underline{I}}^\bullet$ of $\underline{\underline{A}}^\bullet$, $(\underline{\underline{A}}^\bullet \xrightarrow{r} \underline{\underline{I}}^\bullet \xrightarrow{f} \underline{\underline{J}}^\bullet \xleftarrow{s} \underline{\underline{B}}^\bullet) \mapsto (\underline{\underline{I}}^\bullet \xrightarrow{f} \underline{\underline{J}}^\bullet)$ where r,s are the standard resolutions, are inverses as category equivalences.

The category $\mathscr{C}$ is, in fact, isomorphic to the derived category $D^b(X)$, which is more commonly represented in the following form:

<u>Definition</u>: The <u>bounded</u> <u>derived</u> <u>category</u> of X, $D^b(X)$, has as objects bounded complexes of sheaves on X. A morphism $\underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet$ in $D^b(X)$ is represented by a pair of maps in $K^b(X)$, $\underline{\underline{A}}^\bullet \to \underline{\underline{C}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet$ for some $\underline{\underline{C}}^\bullet$ in $K^b(X)$, the second map being a quasi-isomorphism. Two representations $\underline{\underline{A}}^\bullet \to \underline{\underline{C}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet$ and $\underline{\underline{A}}^\bullet \to \underline{\underline{D}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet$ are iden-

tified if a commutative diagram in $K^b(X)$ can be constructed of the following form:

$$\begin{tikzcd}
& \underline{\underline{C}}^{\bullet} \arrow[d, "qi"] & \\
\underline{\underline{A}}^{\bullet} \arrow[ur] \arrow[r] \arrow[dr] & \underline{\underline{E}}^{\bullet} & \underline{\underline{B}}^{\bullet} \arrow[ul, "qi"'] \arrow[l, "qi"'] \arrow[dl, "qi"] \\
& \underline{\underline{D}}^{\bullet} \arrow[u, "qi"'] &
\end{tikzcd}$$

An easy application of lemma 2.1.4 shows that this is an equivalence relation.

Two morphisms $\underline{\underline{A}}^{\bullet} \xrightarrow{f} \underline{\underline{D}}^{\bullet} \xleftarrow[qi]{\alpha} \underline{\underline{B}}^{\bullet}$ and $\underline{\underline{B}}^{\bullet} \xrightarrow{k} \underline{\underline{E}}^{\bullet} \xleftarrow[qi]{\mu} \underline{\underline{C}}^{\bullet}$ are composed by finding a commutative diagram (lemma 2.1.4)

$$\begin{tikzcd}
& \underline{\underline{F}}^{\bullet} & \\
\underline{\underline{D}}^{\bullet} \arrow[ur] & & \underline{\underline{E}}^{\bullet} \arrow[ul, "qi"'] \\
& \underline{\underline{B}}^{\bullet} \arrow[ul, "qi"] \arrow[ur] &
\end{tikzcd}$$

and taking the map $\begin{tikzcd} \underline{\underline{A}}^{\bullet} \arrow[r] \arrow[rr, bend left] & \underline{\underline{D}}^{\bullet} \arrow[r] & \underline{\underline{F}}^{\bullet} & \underline{\underline{E}}^{\bullet} \arrow[l, "qi"'] & \underline{\underline{C}}^{\bullet} \arrow[l, "qi"'] \arrow[ll, bend right, "qi"'] \end{tikzcd}$. Suppose two different squares were formed:

$$\begin{tikzcd}
 & \underline{\underline{F}}^{\bullet} & \\
\underline{\underline{D}}^{\bullet} \arrow[ur, "g"] \arrow[dr, "g'", "qi"'] & \underline{\underline{B}}^{\bullet} \arrow[l, "\alpha"', "qi"] \arrow[r, "k"] & \underline{\underline{E}}^{\bullet} \arrow[ul, "\gamma"', "qi"] \\
 & \underline{\underline{F}}^{\bullet\prime} \arrow[ur, "\gamma'"] &
\end{tikzcd}$$

Let $\lambda : \underline{\underline{E}}^{\bullet} \to \underline{\underline{I}}^{\bullet}$ be an injective resolution. Then we have quasi-isomorphisms $\beta : \underline{\underline{F}}^{\bullet} \to \underline{\underline{I}}^{\bullet}$ and $\beta' : \underline{\underline{F}}^{\bullet\prime} \to \underline{\underline{I}}^{\bullet}$ where $\beta\gamma = \beta'\gamma' = \lambda$ (apply lemma 2.1.4 to γ and λ getting a square of maps with a fourth element $\underline{\underline{R}}^{\bullet}$, and invert the map $\underline{\underline{I}}^{\bullet} \xrightarrow{qi} \underline{\underline{R}}^{\bullet}$ by lemma 2.1.6). Then $\beta g\alpha = \beta\gamma k = \lambda k = \beta'\gamma' k = \beta' g'\alpha$, so $(\beta g - \beta' g')\alpha = 0$. By lemma 2.1.7 (i), then, $\beta g = \beta' g'$ so we have a commutative diagram

$$\begin{tikzcd}
 & & \underline{\underline{F}}^{\bullet} \arrow[d, "\beta"] & & \\
\underline{\underline{A}}^{\bullet} \arrow[r, "f"] & \underline{\underline{D}}^{\bullet} \arrow[ur, "g"] \arrow[r, "\beta g = \beta' g'"] \arrow[dr, "g'"'] & \underline{\underline{I}}^{\bullet} & \underline{\underline{E}}^{\bullet} \arrow[l, "\lambda"'] \arrow[ul, "\gamma"'] \arrow[dl, "\gamma'"] & \underline{\underline{C}}^{\bullet} \arrow[l, "\mu"'] \\
 & & \underline{\underline{F}}^{\bullet\prime} \arrow[u, "\beta'"'] & &
\end{tikzcd}$$

This shows that the composition does not depend on the square chosen. In a similar manner, it can be shown that the composition of two $D^b(X)$ morphisms is independent of the particular representations $\underline{\underline{A}}^\bullet \longrightarrow \underline{\underline{B}}^\bullet \xleftarrow{qi} \underline{\underline{C}}^\bullet$ chosen, and that composition is associative.

There is a functor $F : \mathscr{C} \to D^b(X)$ where $F(\underline{\underline{A}}^\bullet) = \underline{\underline{A}}^\bullet$ and

$$(\underline{\underline{A}}^\bullet \xrightarrow{qi} \underline{\underline{I}}^\bullet \longrightarrow \underline{\underline{J}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet) \longmapsto (\begin{tikzcd} \underline{\underline{A}}^\bullet \arrow[r, "qi"] \arrow[rr, bend left] & \underline{\underline{I}}^\bullet \arrow[r] & \underline{\underline{J}}^\bullet & \underline{\underline{B}}^\bullet \arrow[l, "qi"'] \end{tikzcd}) \quad .$$

<u>Proposition 2.2.1</u>: F is an isomorphism of categories.

<u>Proof</u>: Given $\underline{\underline{A}}^\bullet \longrightarrow \underline{\underline{D}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet \in \operatorname{Hom}_{D^b(X)}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet)$, let $\underline{\underline{A}}^\bullet \to \underline{\underline{I}}^\bullet$ be an injective resolution, and construct a diagram

$$\begin{tikzcd} \underline{\underline{A}}^\bullet \arrow[r] \arrow[d, "qi"] & \underline{\underline{D}}^\bullet \arrow[d, "qi"] \\ \underline{\underline{I}}^\bullet \arrow[r] & \underline{\underline{K}}^\bullet \end{tikzcd}$$

As usual, we can assume $\underline{\underline{K}}^\bullet$ is injective. Then

$$F(\begin{tikzcd} \underline{\underline{A}}^\bullet \arrow[r, "qi"] & \underline{\underline{I}}^\bullet \arrow[r] & \underline{\underline{K}}^\bullet & \underline{\underline{D}}^\bullet \arrow[l, "qi"'] & \underline{\underline{B}}^\bullet \arrow[l, "qi"'] \arrow[lll, bend right, "qi"'] \end{tikzcd}) \text{ is } \begin{tikzcd} \underline{\underline{A}} \arrow[r, "qi"] \arrow[rr, bend left] & \underline{\underline{I}}^\bullet \arrow[r] & \underline{\underline{K}}^\bullet & \underline{\underline{D}}^\bullet \arrow[l, "qi"'] & \underline{\underline{B}}^\bullet \arrow[l, "qi"'] \arrow[lll, bend right, "qi"'] \end{tikzcd}$$

which is equal to $\underline{\underline{A}}^{\bullet} \longrightarrow \underline{\underline{D}}^{\bullet} \xleftarrow{qi} \underline{\underline{B}}^{\bullet}$ because of the diagram

$$\begin{tikzcd}
 & & \underline{\underline{D}}^{\bullet} \arrow[d, "qi"] & & \\
\underline{\underline{A}}^{\bullet} \arrow[urr] \arrow[r] \arrow[dr] & \underline{\underline{I}}^{\bullet} \arrow[r] & \underline{\underline{K}}^{\bullet} & \underline{\underline{D}}^{\bullet} \arrow[l, "qi"'] & \underline{\underline{B}}^{\bullet} \arrow[ull, "qi"'] \arrow[l, "qi"'] \arrow[dl, "qi"] \\
 & \underline{\underline{I}}^{\bullet} \arrow[dr] & & \underline{\underline{D}}^{\bullet} \arrow[dl, "qi"'] & \\
 & & \underline{\underline{K}}^{\bullet} \arrow[uu, "id"'] & &
\end{tikzcd}$$

Now assume we have the two maps $\underline{\underline{A}}^{\bullet} \xrightarrow[qi]{r} \underline{\underline{I}}^{\bullet} \overset{f}{\underset{g}{\rightrightarrows}} \underline{\underline{J}}^{\bullet} \xleftarrow[qi]{s} \underline{\underline{B}}^{\bullet}$ in $\mathscr{C}$ that give the same map under F, i.e., there is a diagram

$$\begin{tikzcd}
 & & \underline{\underline{J}}^{\bullet} \arrow[dd, "\alpha"', "qi"] & \\
 & \underline{\underline{I}}^{\bullet} \arrow[ur, "f"] & & \\
\underline{\underline{A}}^{\bullet} \arrow[ur, "r", "qi"'] \arrow[rr, "\mu"] \arrow[dr, "r", "qi"'] & & \underline{\underline{K}}^{\bullet} & \underline{\underline{B}}^{\bullet} \arrow[uul, "s", "qi"'] \arrow[l, "\beta"', "qi"] \arrow[ddl, "s", "qi"'] \\
 & \underline{\underline{I}}^{\bullet} \arrow[dr, "g"] & & \\
 & & \underline{\underline{J}}^{\bullet} \arrow[uu, "\gamma", "q"'] &
\end{tikzcd}$$

We can assume $\underline{\underline{K}}^{\bullet}$ is injective. Since there is a unique map $\underline{\underline{J}}^{\bullet} \to \underline{\underline{K}}^{\bullet}$ factoring β through s, we have $\alpha = \gamma$. Then

$\alpha fr = \alpha gr = \mu$, so $fr = gr$ since α is invertible, and hence $f = g$ by lemma 2.1.7 (i). Then $F : \mathrm{Hom}_{\mathscr{C}}(\underline{\underline{A}}^{\bullet}, \underline{\underline{B}}^{\bullet}) \to \mathrm{Hom}_{D^b(X)}(\underline{\underline{A}}^{\bullet}, \underline{\underline{B}}^{\bullet})$ is bijective, so F is an isomorphism of categories.

<u>Theorem 2.2.2</u>: Given $\tilde{f}, \tilde{g} \in \mathrm{Hom}_{D^b(X)}(\underline{\underline{A}}^{\bullet}, \underline{\underline{B}}^{\bullet})$, representations can be found of each that use the same middle object and quasi-isomorphism: $\tilde{f}, \tilde{g} : \underline{\underline{A}}^{\bullet} \longrightarrow \underline{\underline{D}}^{\bullet} \xleftarrow{qi} \underline{\underline{B}}^{\bullet}$. $\tilde{f} + \tilde{g}$ can then be formed by adding the maps $\underline{\underline{A}}^{\bullet} \to \underline{\underline{D}}^{\bullet}$, and leaving the quasi-isomorphism untouched (where addition is defined so as to correspond to addition in $\mathscr{C}$).

<u>Proof</u>: The first statement is clear by proposition 2.2.1. To see the second statement, let $\tilde{f}$ be given by $\underline{\underline{A}}^{\bullet} \xrightarrow{f} \underline{\underline{D}}^{\bullet} \xleftarrow[qi]{\alpha} \underline{\underline{B}}^{\bullet}$ and g be given by $\underline{\underline{A}} \xrightarrow{g} \underline{\underline{D}}^{\bullet} \xleftarrow[qi]{\alpha} \underline{\underline{B}}^{\bullet}$. Form a diagram

$$\begin{tikzcd} \underline{\underline{A}}^{\bullet} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\beta"', "qi"] & \underline{\underline{D}}^{\bullet} \arrow[d, "\gamma"', "qi"] & \underline{\underline{B}}^{\bullet} \arrow[l, "\alpha"', "qi"] \\ \underline{\underline{I}}^{\bullet} \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & \underline{\underline{J}}^{\bullet} & \end{tikzcd}$$

where $\underline{\underline{I}}^{\bullet}$ and $\underline{\underline{J}}^{\bullet}$ are injective complexes, $\gamma\tilde{f} = f'\beta$, and $\gamma\tilde{g} = g'\beta$. Then $\tilde{f}$ and $\tilde{g}$ correspond in $\mathscr{C}$ to $\underline{\underline{A}}^{\bullet} \xrightarrow{\beta} \underline{\underline{I}}^{\bullet} \xrightarrow{f'} \underline{\underline{J}}^{\bullet} \xleftarrow{\gamma\alpha} \underline{\underline{B}}^{\bullet}$ and $\underline{\underline{A}}^{\bullet} \xrightarrow{\beta} \underline{\underline{I}}^{\bullet} \xrightarrow{g'} \underline{\underline{J}}^{\bullet} \xleftarrow{\gamma\alpha} \underline{\underline{B}}^{\bullet}$, respectively, so $\tilde{f} + \tilde{g}$ is represented by $\underline{\underline{A}}^{\bullet} \xrightarrow{(f'+g')\beta} \underline{\underline{J}}^{\bullet} \xleftarrow{\gamma\alpha} \underline{\underline{B}}^{\bullet}$. Since $(f'+g')\beta = \gamma(f+g)$, it follows that $\widetilde{f+g}$ is also represented by

$\underline{\underline{A}}^\bullet \xrightarrow{f+g} \underline{\underline{D}}^\bullet \xleftarrow[q^i]{\alpha} \underline{\underline{B}}^\bullet$.

There is a functor $\mathscr{F} : K^b(X) \to D^b(X)$ that takes an object to itself, and takes $\underline{\underline{A}}^\bullet \xrightarrow{f} \underline{\underline{B}}^\bullet$ to $\underline{\underline{A}}^\bullet \xrightarrow{f} \underline{\underline{B}}^\bullet \xleftarrow{id} \underline{\underline{B}}^\bullet$. It is <u>not</u> a faithful functor.

<u>Theorem 2.2.3</u>: If $f,g : \underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet$ are maps in $K^b(X)$, then they represent the same maps in $D^b(X)$ (i.e., $\mathscr{F}(f) = \mathscr{F}(g)$) if and only if there is a quasi-isomorphism $\alpha : \underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet$ such that $\alpha f = \alpha g$. This is also true if and only if there is a quasi-isomorphism $j : \underline{\underline{R}}^\bullet \to \underline{\underline{A}}^\bullet$ such that $fj = gj$.

<u>Proof</u>: Given $f : \underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet$ in $K^b(X)$, there exists a quasi-isomorphism $\alpha : \underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet$ such that $\alpha f = 0$ if and only if there is a commutative diagram

$$\begin{tikzcd}
 & \underline{\underline{B}}^\bullet \arrow[d, "\alpha"', "q^i"] & \\
\underline{\underline{A}}^\bullet \arrow[ur, "f"] \arrow[r, "0"] \arrow[dr, "0"'] & \underline{\underline{C}}^\bullet & \underline{\underline{B}}^\bullet \arrow[ul, "id"'] \arrow[l, "\alpha"', "q^i"] \arrow[dl, "id"] \\
 & \underline{\underline{B}}^\bullet \arrow[u, "\alpha", "q^i"'] &
\end{tikzcd}$$

i.e., if and only if f represents the 0 map in $D^b(X)$. This shows the first statement.

To show the second statement, let $\underline{\underline{R}}^\bullet \xrightarrow{j} \underline{\underline{A}}^\bullet \xrightarrow{f} \underline{\underline{B}}^\bullet$ be a diagram in $K^b(X)$ with $fj = 0$ and j a quasi-isomorphism. Then for an injective resolution $i : \underline{\underline{B}}^\bullet \to \underline{\underline{I}}^\bullet$, $ifj = 0$, so by lemma 2.1.7 (i), $if = 0$, hence f represents 0 in $D^b(X)$ by the first part of the theorem. Conversely, if f represents 0 in $D^b(X)$, then there is a quasi-isomorphism $\alpha : \underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet$ with $\alpha f = 0$. Taking a projective resolution $j : \underline{\underline{P}}^\bullet \to \underline{\underline{A}}^\bullet$, we have $\alpha fj = 0$, so by lemma 2.1.7 (ii), $fj = 0$.

Remarks: $D^b(X)$ can also be described as having the same objects, but morphisms being of the form $\underline{\underline{A}}^\bullet \xleftarrow{qi} \underline{\underline{C}}^\bullet \longrightarrow \underline{\underline{B}}^\bullet$, two maps $\underline{\underline{A}}^\bullet \xleftarrow{qi} \underline{\underline{C}}^\bullet \longrightarrow \underline{\underline{B}}^\bullet$, $\underline{\underline{A}} \xleftarrow{qi} \underline{\underline{D}} \longrightarrow \underline{\underline{B}}^\bullet$ being identified if there is a diagram

$$\begin{tikzcd}
 & \underline{\underline{C}}^\bullet \arrow[dl, "qi"'] \arrow[dr] & \\
\underline{\underline{A}}^\bullet & \underline{\underline{E}}^\bullet \arrow[l, "qi"'] \arrow[u, "qi"'] \arrow[r] \arrow[d, "qi"] & \underline{\underline{B}}^\bullet \\
 & \underline{\underline{D}}^\bullet \arrow[ul, "qi"] \arrow[ur] &
\end{tikzcd}$$

The same construction can be made on VS, resulting in the derived category $D^b(VS)$. However, since all elements of VS are injective, we have that $D^b(VS)$ is isomorphic to $K^b(VS)$.

§2.3 Derived Functors

Let $F : S(X) \to S(Y)$ or $F : S(X) \to VS$ be a covariant functor, and $F^\bullet$ the induced functor on $K^b(X)$. $F^\bullet$ in general does not extend in a simple way to a functor between derived categories, because $F^\bullet$ does not necessarily send quasi-isomorphisms to quasi-isomorphisms, so it is not clear what morphism to associate to $\underline{\underline{A}}^\bullet \longrightarrow \underline{\underline{C}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet$. Instead, one can define the derived functor of F, $R^\bullet F : D^b(X) \to D^b(Y)$ (or $R^\bullet F : D^b(X) \to D^b(VS) = K^b(VS)$), by $R^\bullet F(\underline{\underline{A}}^\bullet) = F^\bullet(\underline{\underline{I}}^\bullet)$, where $\underline{\underline{A}}^\bullet \xrightarrow{qi} \underline{\underline{I}}^\bullet$ is the standard injective resolution of $\underline{\underline{A}}^\bullet$. The morphisms in $D^b(X)$ from $\underline{\underline{A}}^\bullet$ to $\underline{\underline{B}}^\bullet$ are given by diagrams $\underline{\underline{A}}^\bullet \xrightarrow{qi} \underline{\underline{I}}^\bullet \xrightarrow{f} \underline{\underline{J}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet$ where $\underline{\underline{A}}^\bullet \xrightarrow{qi} \underline{\underline{I}}^\bullet$ and $\underline{\underline{B}}^\bullet \xrightarrow{qi} \underline{\underline{J}}^\bullet$ are the standard injective resolutions, so we can define $R^\bullet F(\underline{\underline{A}}^\bullet \xrightarrow{qi} \underline{\underline{I}}^\bullet \xrightarrow{f} \underline{\underline{J}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet)$ to be the morphism in $D^b(Y)$ represented by $F^\bullet(f) \in \mathrm{Hom}_{K^b(X)}(F^\bullet\underline{\underline{I}}^\bullet, F^\bullet\underline{\underline{J}}^\bullet)$.

Although we need to choose a fixed resolution for each element of $K^b(X)$ to make the functor $R^\bullet F$ well-defined (e.g., the standard resolution), the following theorem will show that the choice of fixed injective resolutions is arbitrary, and that in applications where one is only interested in objects defined up to quasi-isomorphism, it is not necessary to choose fixed resolutions at all.

<u>Theorem 2.3.1</u>: Given two injective resolutions $\underline{\underline{A}}^\bullet \xrightarrow{\alpha} \underline{\underline{I}}^\bullet$, $\underline{\underline{A}}^\bullet \xrightarrow{\alpha'} \underline{\underline{I}}'^\bullet$ and a functor F defined on $S(X)$, there is a canonical quasi-isomorphism $q_{\alpha,\alpha'} : F^\bullet\underline{\underline{I}}^\bullet \to F^\bullet\underline{\underline{I}}'^\bullet$. Given a morphism

$\tilde{f} \in \mathrm{Hom}_{D^b(X)}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet)$ and injective resolutions $\underline{\underline{B}}^\bullet \xrightarrow{\beta} \underline{\underline{J}}^\bullet$, $\underline{\underline{B}}^\bullet \xrightarrow{\beta'} \underline{\underline{J}}'^\bullet$, let $\tilde{f}$ be given by $\underline{\underline{A}}^\bullet \xrightarrow{\alpha} \underline{\underline{I}}^\bullet \xrightarrow{f} \underline{\underline{J}}^\bullet \xleftarrow{\beta} \underline{\underline{B}}^\bullet$ and by $\underline{\underline{A}}^\bullet \xrightarrow{\alpha'} \underline{\underline{I}}'^\bullet \xrightarrow{f'} \underline{\underline{J}}'^\bullet \xleftarrow{\beta'} \underline{\underline{B}}^\bullet$. Then the following diagram commutes:

$$\begin{tikzcd}
F^\bullet \underline{\underline{I}}^\bullet \arrow[r, "q_{\alpha,\alpha'}"] \arrow[d, "F^\bullet f"'] & F^\bullet \underline{\underline{I}}'^\bullet \arrow[d, "F^\bullet f'"] \\
F^\bullet \underline{\underline{J}}^\bullet \arrow[r, "q_{\beta,\beta'}"] & F^\bullet \underline{\underline{J}}'^\bullet
\end{tikzcd} .$$

<u>Proof</u>: There is a unique quasi-isomorphism $h : \underline{\underline{I}}^\bullet \to \underline{\underline{I}}'^\bullet$ in $K^b(X)$ such that $h\alpha = \alpha'$. Then $F^\bullet h$ is invertible since h is invertible, hence $F^\bullet h$ is a quasi-isomorphism. Let $q_{\alpha,\alpha'}$ be $F^\bullet h$. The second statement in the theorem follows from the fact that $f'h = kf$ where $k : \underline{\underline{J}}^\bullet \to \underline{\underline{J}}'^\bullet$ is the unique map such that $k\beta = \beta'$.

It should be noted that $\underline{\underline{H}}^i$, $\mathbb{H}^i$, and $\mathbb{H}^i_c$ extend to functors on $D^b(X)$: $\underline{\underline{H}}^i$ will associate to $\underline{\underline{A}}^\bullet \longrightarrow \underline{\underline{B}}^\bullet \xleftarrow{qi} \underline{\underline{C}}^\bullet$ the sheaf map $\underline{\underline{H}}^i\underline{\underline{A}}^\bullet \to \underline{\underline{H}}^i\underline{\underline{C}}^\bullet$ acquired from the induced diagram $\underline{\underline{H}}^i\underline{\underline{A}}^\bullet \longrightarrow \underline{\underline{H}}^i\underline{\underline{B}}^\bullet \xleftarrow{\cong} \underline{\underline{H}}^i\underline{\underline{C}}^\bullet$ by inverting the isomorphism. Since quasi-isomorphisms induce isomorphisms on hypercohomology, the functors on $\mathbb{H}^i$ and $\mathbb{H}^i_c$ can be defined in the same way.

If $F : S(X) \to S(Y)$, we write $R^iF : D^b(X) \to S(Y)$ for the functor $\underline{\underline{H}}^i \circ R^{\bullet}F$. If $F : S(X) \to VS$, then $R^iF : D^b(X) \to VS$ will be the functor $H^i \circ R^{\bullet}F$.

<u>Remark</u>: When working with the functor $R^{\bullet}f_!$, one generally wants to consider only cell maps satisfying further conditions given in §3.3. These conditions are in particular needed to have $R^{\bullet}f_!$ correspond to the functor $R^{\bullet}f_!$ used in the conventional theory of derived categories.

It follows immediately from the definition that $f^* : K^b(Y) \to K^b(X)$ preserves quasi-isomorphisms, so we can extend f^* to a functor on $D^b(Y)$ directly--let $f^*(A^{\bullet})$ be the same as before and $f^*(\underline{\underline{A}}^{\bullet} \xrightarrow{h} \underline{\underline{C}}^{\bullet} \xleftarrow[qi]{\alpha} \underline{\underline{B}}^{\bullet})$ be $f^*\underline{\underline{A}}^{\bullet} \xrightarrow{f^*h} f^*\underline{\underline{C}}^{\bullet} \xleftarrow[qi]{f^*\alpha} \underline{\underline{B}}^{\bullet}$. It is clear that if two diagrams $\underline{\underline{A}}^{\bullet} \to \underline{\underline{C}}^{\bullet} \xleftarrow[qi]{} \underline{\underline{B}}^{\bullet}$ and $\underline{\underline{A}}^{\bullet} \to \underline{\underline{D}}^{\bullet} \xleftarrow[qi]{} \underline{\underline{B}}^{\bullet}$ represent the same morphism in $D^b(X)$ then their images under f^* represent the same map, and that $f^*(h \circ k) = (f^*h) \circ (f^*k)$.

Given $\tilde{r} : \underline{\underline{A}}^{\bullet} \to \underline{\underline{B}}^{\bullet}$ a map in $D^b(X)$, $\tilde{r}$ given in $\mathscr{C}$ by $\underline{\underline{A}}^{\bullet} \xrightarrow[qi]{\alpha} \underline{\underline{I}}^{\bullet} \xrightarrow{r} \underline{\underline{J}}^{\bullet} \xleftarrow[qi]{\beta} \underline{\underline{B}}^{\bullet}$, α and β being standard injective resolutions, it can be checked that $\beta\tilde{r} = r\alpha$ in $D^b(X)$ where β, r , and α are taken to be the morphisms in $D^b(X)$ they represent. We then have a commutative diagram in $D^b(X)$,

$$\begin{tikzcd} f^*\underline{\underline{A}}^\bullet \arrow[r, "f^*\alpha", "qi"'] \arrow[d, "f^*\tilde{r}"'] & f^*\underline{\underline{I}}^\bullet = R^\bullet f^*\underline{\underline{A}}^\bullet \arrow[d, "f^*r"] \\ f^*\underline{\underline{B}}^\bullet \arrow[r, "f^*\beta", "qi"'] & f^*\underline{\underline{J}}^\bullet = R^\bullet f^*\underline{\underline{B}}^\bullet \end{tikzcd} .$$

Hence the functors f^* and $R^\bullet f^*$ are the same, up to a natural transformation given by the quasi-isomorphisms $f^*\alpha$.

$\otimes$ and $R^\bullet \underline{\underline{Hom}}$

Given two elements $\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet \in D^b(X)$, we would like to define $\underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet \in D^b(X)$ and $R^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet) \in D^b(X)$ in such a way that to each correspondence $\underline{\underline{B}}^\bullet \longmapsto \underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet$, $\underline{\underline{A}}^\bullet \longmapsto \underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet$, $\underline{\underline{B}}^\bullet \longmapsto R^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet)$, and $\underline{\underline{A}}^\bullet \longmapsto R^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet)$, we can in a natural way associate a functor on $D^b(X)$ that maps objects in the prescribed way (the first three covariant, the last one contravariant).

Defining $\underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet$ on $D^b(X)$ is straightforward, once we have the following lemma:

<u>Lemma 2.3.2</u>: A quasi-isomorphism $\underline{\underline{B}}^\bullet \xrightarrow{qi} \underline{\underline{C}}^\bullet$ in $K^b(X)$ induces quasi-isomorphisms $\underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet \xrightarrow{qi} \underline{\underline{A}}^\bullet \otimes \underline{\underline{C}}^\bullet$ and $\underline{\underline{B}}^\bullet \otimes \underline{\underline{A}}^\bullet \xrightarrow{qi} \underline{\underline{C}}^\bullet \otimes \underline{\underline{A}}^\bullet$ in $K^b(X)$.

Proof: A q.i. (i.e., a quasi-isomorphism) $B^{\bullet} \xrightarrow{qi} C^{\bullet}$ between complexes of vector spaces induces a q.i. $A \otimes B^{\bullet} \xrightarrow{qi} A \otimes C^{\bullet}$ for A a single vector space, so it follows immediately that a chain map $\underline{\underline{B}}^{\bullet} \xrightarrow{qi} \underline{\underline{C}}^{\bullet}$ between chain complexes of sheaves induces a q.i. $\underline{\underline{A}} \otimes \underline{\underline{B}}^{\bullet} \xrightarrow{qi} \underline{\underline{A}} \otimes \underline{\underline{C}}^{\bullet}$, $\underline{\underline{A}}$ a sheaf. Given $\underline{\underline{B}}^{\bullet} \xrightarrow{qi} \underline{\underline{C}}^{\bullet}$ and $\underline{\underline{A}}^{\bullet} \in K^b(X)$, the map being represented by a chain map $f^{\bullet}$, the induced map $\underline{\underline{A}}^{\bullet} \otimes \underline{\underline{B}}^{\bullet} \longrightarrow \underline{\underline{A}}^{\bullet} \otimes \underline{\underline{C}}^{\bullet}$ is acquired by forming the double complex map $\underline{\underline{A}}^{i} \otimes \underline{\underline{B}}^{j} \xrightarrow{id \otimes f^{j}} \underline{\underline{A}}^{i} \otimes \underline{\underline{C}}^{j}$ and taking the associated map on single complexes. But since for a fixed i, $\underline{\underline{A}}^{i} \otimes \underline{\underline{B}}^{\bullet} \xrightarrow{id \otimes f^{\bullet}} \underline{\underline{A}}^{i} \otimes \underline{\underline{C}}^{\bullet}$ is a q.i. it follows from Theorem 1.2.3 that $\underline{\underline{A}}^{\bullet} \otimes \underline{\underline{B}}^{\bullet} \to \underline{\underline{A}}^{\bullet} \otimes \underline{\underline{C}}^{\bullet}$ is a q.i. The proof that $\underline{\underline{B}}^{\bullet} \xrightarrow{qi} \underline{\underline{C}}^{\bullet}$ induces a q.i. $\underline{\underline{B}}^{\bullet} \otimes \underline{\underline{A}}^{\bullet} \xrightarrow{qi} \underline{\underline{C}}^{\bullet} \otimes \underline{\underline{A}}^{\bullet}$ is the same.

We can now define $\underline{\underline{A}}^{\bullet} \otimes \underline{\underline{B}}^{\bullet}$ in $D^b(X)$ to be the same as in $K^b(X)$, and have the map $\underline{\underline{B}}^{\bullet} \longrightarrow \underline{\underline{D}}^{\bullet} \xleftarrow{qi} \underline{\underline{C}}^{\bullet}$ induce a map $\underline{\underline{A}}^{\bullet} \otimes \underline{\underline{B}}^{\bullet} \longrightarrow \underline{\underline{A}}^{\bullet} \otimes \underline{\underline{D}}^{\bullet} \xleftarrow{qi} \underline{\underline{A}}^{\bullet} \otimes C^{\bullet}$. Similarly, $\underline{\underline{A}}^{\bullet} \longrightarrow \underline{\underline{D}}^{\bullet} \xleftarrow{qi} \underline{\underline{C}}$ will induce $\underline{\underline{A}}^{\bullet} \otimes \underline{\underline{B}}^{\bullet} \longrightarrow \underline{\underline{D}}^{\bullet} \otimes \underline{\underline{B}}^{\bullet} \xleftarrow{qi} \underline{\underline{C}}^{\bullet} \otimes \underline{\underline{B}}^{\bullet}$. As with f^{*}, it is easy to see that this is well-defined, i.e., different representations of the same $D^b(X)$ map give representations of the same map, that this process is compatible with composing $D^b(X)$ maps, and that the two functors $\underline{\underline{A}}^{\bullet} \longrightarrow \underline{\underline{A}}^{\bullet} \otimes \underline{\underline{B}}^{\bullet}$ and $\underline{\underline{B}}^{\bullet} \longrightarrow \underline{\underline{A}}^{\bullet} \otimes \underline{\underline{B}}^{\bullet}$ are the same (up to q.i. as their derived functors.

It will be necessary to do a derived functor construction on $\underline{\underline{Hom}}^{\bullet}$, as the two $\underline{\underline{Hom}}^{\bullet}$ functors do not in general preserve q.i.'s.

Definition: $R^{\bullet}\underline{\underline{Hom}}(\underline{\underline{A}}^{\bullet}, \underline{\underline{B}}^{\bullet}) = \underline{\underline{Hom}}^{\bullet}(\underline{\underline{A}}^{\bullet}, \underline{\underline{I}}^{\bullet})$ where $\underline{\underline{B}}^{\bullet} \to \underline{\underline{I}}^{\bullet}$ is the standard injective resolution.

This defines a natural covariant functor $R^{\bullet}\underline{\underline{Hom}}(\underline{\underline{A}}^{\bullet},\cdot)$ on $D^b(X)$, where $\underline{\underline{B}}^{\bullet} \xrightarrow{qi} \underline{\underline{I}}^{\bullet} \xrightarrow{f} \underline{\underline{J}}^{\bullet} \xleftarrow{qi} \underline{\underline{C}}^{\bullet}$ gives rise to

$\underline{\underline{Hom}}^{\bullet}(\underline{\underline{A}}^{\bullet},\underline{\underline{I}}^{\bullet}) \xrightarrow{f_{\#}} \underline{\underline{Hom}}^{\bullet}(\underline{\underline{A}}^{\bullet},\underline{\underline{J}}^{\bullet})$, $f_{\#}$ induced by F. As with previous derived functors, given two injective resolutions $\underline{\underline{B}}^{\bullet} \to \underline{\underline{I}}^{\bullet}$ and $\underline{\underline{B}}^{\bullet} \to \underline{\underline{I}}'^{\bullet}$, there is a canonical quasi-isomorphism

$\underline{\underline{Hom}}(\underline{\underline{A}}^{\bullet},\underline{\underline{I}}^{\bullet}) \xrightarrow{qi} \underline{\underline{Hom}}(\underline{\underline{A}}^{\bullet},\underline{\underline{I}}'^{\bullet})$, and these quasi-isomorphisms give a natural transformation between functors $\underline{\underline{B}}^{\bullet} \to R^{\bullet}\underline{\underline{Hom}}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet})$ defined in terms of different fixed choices of injective resolutions.

Theorem 2.2.3: (i) If $\underline{\underline{I}}^{\bullet}$ is an injective complex and $\underline{\underline{A}}^{\bullet} \to \underline{\underline{C}}^{\bullet}$ is a quasi-isomorphism in $K^b(X)$, then the induced map $\underline{\underline{Hom}}^{\bullet}(\underline{\underline{C}}^{\bullet},\underline{\underline{I}}^{\bullet}) \to \underline{\underline{Hom}}^{\bullet}(\underline{\underline{A}}^{\bullet},\underline{\underline{I}}^{\bullet})$ is a quasi-isomorphism.

(ii) If $\underline{\underline{P}}^{\bullet}$ is a projective complex, then the induced map $\underline{\underline{Hom}}^{\bullet}(\underline{\underline{P}}^{\bullet},\underline{\underline{A}}^{\bullet}) \to \underline{\underline{Hom}}^{\bullet}(\underline{\underline{P}}^{\bullet},\underline{\underline{C}}^{\bullet})$ is a quasi-isomorphism.

Proof: (i) We may assume that $f^{\bullet} : \underline{\underline{A}}^{\bullet} \to \underline{\underline{C}}^{\bullet}$ is a chain map, and show that the induced chain map is a q.i. By Theorem 1.2.3, it will suffice to show that $f^{\bullet} : \underline{\underline{A}}^{\bullet} \xrightarrow{qi} \underline{\underline{C}}^{\bullet}$ induces a q.i. $f^{\#} : \underline{\underline{Hom}}^{\bullet}(\underline{\underline{C}}^{\bullet},\underline{\underline{I}}) \to \underline{\underline{Hom}}^{\bullet}(\underline{\underline{A}}^{\bullet},\underline{\underline{I}})$ for $\underline{\underline{I}}$ a single injective sheaf. Furthermore, since $\underline{\underline{I}} = \oplus [\sigma_i]^{V_i}$ and $\underline{\underline{Hom}}^{\bullet}$ and the induced chain map are compatible with this decomposition, we may assume that $\underline{\underline{I}} = [\sigma]^V$.

For $\tau \nleq \sigma$, $\underline{\underline{Hom}}^{\bullet}(\underline{\underline{C}}^{\bullet},[\sigma]^V)(\tau)$ and $\underline{\underline{Hom}}^{\bullet}(\underline{\underline{A}}^{\bullet},[\sigma]^V)(\tau)$ are both the 0-complex, so $f^{\#}$ is a q.i. over these cells. Let $\tau \leq \sigma$. Then

$\underline{\underline{\text{Hom}}}^{-i}(\underline{\underline{A}}^\bullet,[\sigma]^V)(\tau)$ = all sets of compatible maps $\{h_\gamma : \underline{\underline{A}}^i(\gamma) \to V | \tau \le \gamma \le \sigma\}$. It's clear there is a unique such set for every element of $\text{Hom}(\underline{\underline{A}}^i(\sigma),V)$ (given $\alpha \in \text{Hom}(\underline{\underline{A}}^i(\sigma),V)$, let $h_\gamma = \alpha\, p_{\gamma,\sigma}$), so we have that $\text{Hom}(\underline{\underline{A}}^i(\sigma),V) \cong \underline{\underline{\text{Hom}}}^{-i}(\underline{\underline{A}}^\bullet,[\sigma]^V)(\tau)$. It's also easily checked that these isomorphisms are compatible with the boundary maps in $\underline{\underline{\text{Hom}}}^\bullet(\underline{\underline{A}}^\bullet,[\sigma]^V)(\tau)$ and the chain map $f^\#$ over τ, i.e. the diagram $f^\#_\tau : \underline{\underline{\text{Hom}}}^\bullet(\underline{\underline{C}}^\bullet,[\sigma]^V)(\tau) \to \underline{\underline{\text{Hom}}}^\bullet(\underline{\underline{A}}^\bullet,[\sigma]^V)(\tau)$ is isomorphic to $\text{Hom}^\bullet(\underline{\underline{C}}^\bullet(\sigma),V) \to \text{Hom}^\bullet(\underline{\underline{A}}^\bullet(\sigma),V)$ where all maps are the naturally induced ones. But $\underline{\underline{A}}^\bullet(\sigma) \to \underline{\underline{C}}^\bullet(\sigma)$ being a q.i. implies that this is a q.i. since on the category of vector spaces, $\text{Hom}(\cdot,V)$ is an exact functor.

The proof of (ii) is similar.

Theorem 2.3.3 (ii) gives us,

<u>Corollary 2.3.4</u>: If $\underline{\underline{P}}^\bullet$ is a projective complex, then $R^\bullet\underline{\underline{\text{Hom}}}(\underline{\underline{P}}^\bullet,\underline{\underline{B}}^\bullet) \overset{qi}{=} \underline{\underline{\text{Hom}}}^\bullet(\underline{\underline{P}}^\bullet,\underline{\underline{B}}^\bullet)$.

We now define a contravariant functor $\underline{\underline{A}}^\bullet \longmapsto R^\bullet\underline{\underline{\text{Hom}}}(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)$ by sending the map $\underline{\underline{A}}^\bullet \xrightarrow{f} \underline{\underline{D}}^\bullet \xleftarrow[qi]{\alpha} \underline{\underline{C}}^\bullet$ to the $D^b(X)$ map given by the composition

$\underline{\underline{\text{Hom}}}^\bullet(\underline{\underline{C}}^\bullet,\underline{\underline{I}}^\bullet) \xleftarrow[qi]{\alpha^\#} \underline{\underline{\text{Hom}}}^\bullet(\underline{\underline{D}}^\bullet,\underline{\underline{I}}^\bullet) \xrightarrow{f^\#} \underline{\underline{\text{Hom}}}^\bullet(\underline{\underline{A}}^\bullet,\underline{\underline{I}}^\bullet)$ where $\underline{\underline{B}}^\bullet \to \underline{\underline{I}}^\bullet$ is the standard injective resolution ($\alpha^\#$ is a quasi-isomorphism by theorem 2.3.3 (i)). By the remark at the end of §2.2, two diagrams of the form $\underline{\underline{A}}^\bullet \longrightarrow \underline{\underline{D}}^\bullet \xleftarrow{qi} \underline{\underline{C}}^\bullet$ that represent the same $D^b(X)$ map will induce the same $D^b(X)$ maps when the functor $\underline{\underline{\text{Hom}}}^\bullet(\cdot,\underline{\underline{I}}^\bullet)$ is applied.

We can also form $R^{\bullet}\mathrm{Hom}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}) = \mathrm{Hom}(A^{\bullet},\underline{\underline{I}}^{\bullet})$ for $\underline{\underline{B}}^{\bullet} \to \underline{\underline{I}}^{\bullet}$ the standard injective resolution in the same way. All of the above constructions and theorems for $R^{\bullet}\underline{\underline{\mathrm{Hom}}}$ apply for $R^{\bullet}\mathrm{Hom}$, and the proofs are identical.

We will write $\underline{\underline{\mathrm{Ext}}}^{i}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}) = R^{i}\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}) = \underline{\underline{H}}^{i}(R^{\bullet}\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}))$ and $\mathrm{Ext}^{i}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}) = R^{i}\mathrm{Hom}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}) = H^{i}(R^{\bullet}\mathrm{Hom}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}))$.

Lemma 2.3.5: If $\underline{\underline{I}}^{\bullet}$ is an injective complex, then $\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{\bullet},\underline{\underline{I}}^{\bullet})$ is an injective complex.

Proof: It can be verified that $\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}},[\sigma]^{V}) \cong [\sigma]^{\mathrm{Hom}(\underline{\underline{A}}(\sigma),V)}$. The lemma follows immdediately from this.

Theorem 2.3.6: (i) $R^{i}\Gamma(\underline{\underline{A}}^{\bullet}) \cong \mathbb{H}^{i}(\underline{\underline{A}}^{\bullet})$ and $R^{i}\Gamma_{c}(\underline{\underline{A}}^{\bullet}) \cong \mathbb{H}^{i}_{c}(\underline{\underline{A}}^{\bullet})$

(ii) $R^{\bullet}\mathrm{Hom}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}) \ \underline{qi}\ R^{\bullet}\Gamma(R^{\bullet}\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}))$

(iii) $\mathrm{Ext}^{i}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}) \cong \mathbb{H}^{i}(R^{\bullet}\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}))$

(iv) $\mathrm{Hom}_{D^{b}(X)}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}) \cong \mathrm{Ext}^{0}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet})$

Proof: (i) This is just theorem 1.4.2.

(ii) $R^{\bullet}\Gamma(R^{\bullet}\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet})) \ \underline{\underline{qi}}\ R^{\bullet}\Gamma(\underline{\underline{\mathrm{Hom}}}^{\bullet}(\underline{\underline{A}}^{\bullet},\underline{\underline{I}}^{\bullet}))$ for $\underline{\underline{B}}^{\bullet} \to \underline{\underline{I}}^{\bullet}$ and injective resolution, $\underline{\underline{qi}}\ \Gamma^{\bullet}(\underline{\underline{\mathrm{Hom}}}^{\bullet}(\underline{\underline{A}}^{\bullet},\underline{\underline{I}}^{\bullet}))$ by lemma 2.3.5, $= \mathrm{Hom}^{\bullet}(\underline{\underline{A}}^{\bullet},\underline{\underline{I}}^{\bullet}) \ \underline{\underline{qi}}\ R^{\bullet}\mathrm{Hom}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet})$.

(iii) $\mathrm{Ext}^{i}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}) \cong H^{i}(R^{\bullet}\mathrm{Hom}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet})) \cong H^{i}(R^{\bullet}\Gamma(R^{\bullet}\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet})))$ by (ii), $\cong \mathbb{H}^{i}(R^{\bullet}\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}))$ by (i).

(iv) Given injective resolutions $\underline{\underline{A}}^{\bullet} \to \underline{\underline{I}}^{\bullet}$ and $\underline{\underline{B}}^{\bullet} \to \underline{\underline{J}}^{\bullet}$, we have $\mathrm{Hom}_{D^{b}(X)}(\underline{\underline{A}}^{\bullet},\underline{\underline{B}}^{\bullet}) \cong \mathrm{Hom}_{D^{b}(X)}(\underline{\underline{I}}^{\bullet},\underline{\underline{J}}^{\bullet}) \cong \mathrm{Hom}_{K^{b}(X)}(\underline{\underline{I}}^{\bullet},\underline{\underline{J}}^{\bullet}) \cong$

$\mathrm{Ext}^0(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet)$ by theorem 2.1.3.

2.4 Truncation Functors

We will define two new functors $\tau_{\leq p} : D^b(X) \to D^b(X)$ and $\tau^{\geq p} : D^b(X) \to D^b(X)$ for $p \in \mathbb{Z}$, which will "truncate" the stalk cohomology of elements of $D^b(X)$; specifically, we will have

$$\underline{\underline{H}}^i \tau_{\leq p} \underline{\underline{A}}^\bullet \cong \begin{cases} \underline{\underline{H}}^i \underline{\underline{A}}^\bullet & i \leq p \\ 0 & i > p \end{cases}, \quad \text{and} \quad \underline{\underline{H}}^i \tau^{\geq p} \underline{\underline{A}}^\bullet \cong \begin{cases} \underline{\underline{H}}^i \underline{\underline{A}}^\bullet & i \geq p \\ 0 & i < p \end{cases}$$

These conditions come close to characterizing $\tau_{\leq p}$ and $\tau^{\geq p}$, as the following lemma shows.

Lemma 2.4.1: Let $\tau, \tau' : D^b(X) \to D^b(X)$ be functors with natural transformations $\mathrm{id} \to \tau$, $\mathrm{id} \to \tau'$ (respectively $\tau \to \mathrm{id}$, $\tau' \to \mathrm{id}$) such that $\underline{\underline{H}}^i \tau \underline{\underline{A}}^\bullet = \underline{\underline{H}}^i \tau' \underline{\underline{A}}^\bullet = 0$ for $i < p$ (respectively $i > p$) and $\underline{\underline{A}}^\bullet \to \tau \underline{\underline{A}}^\bullet$, $\underline{\underline{A}}^\bullet \to \tau' \underline{\underline{A}}$ induce isomorphisms $\underline{\underline{H}}^i \underline{\underline{A}}^\bullet \to \underline{\underline{H}}^i \tau \underline{\underline{A}}^\bullet$, $\underline{\underline{H}}^i \underline{\underline{A}}^\bullet \to \underline{\underline{H}}^i \tau' \underline{\underline{A}}^\bullet$ for $i \geq p$ (respectively $i \leq p$). Then τ, τ' are isomorphic as functors, i.e., there is a natural transformation $\tau \to \tau'$ where the maps $\tau \underline{\underline{A}}^\bullet \to \tau' \underline{\underline{A}}^\bullet$ are isomorphisms (in $D^b(X)$).

Proof: To prove the first statement (where there are transormations $\mathrm{id} \to \tau$), apply τ to $\underline{\underline{A}}^\bullet \to \tau' \underline{\underline{A}}^\bullet$, getting a map $\tau \underline{\underline{A}}^\bullet \to \tau\tau' \underline{\underline{A}}$. The stalk cohomology of these sheaves is 0 in degrees $i < p$, and for $i \geq p$ we have the diagram

$$\begin{tikzcd} \underline{\underline{A}}^\bullet \arrow[r] \arrow[d] & \tau\underline{\underline{A}}^\bullet \arrow[d] \\ \tau'\underline{\underline{A}}^\bullet \arrow[r] & \tau\tau'\underline{\underline{A}}^\bullet \end{tikzcd}$$

which gives

$$\begin{tikzcd} \underline{\underline{H}}^i\underline{\underline{A}}^\bullet \arrow[r, "\cong"] \arrow[d, "\cong"'] & \underline{\underline{H}}^i\tau\underline{\underline{A}}^\bullet \arrow[d] \\ \underline{\underline{H}}^i\tau'\underline{\underline{A}}^\bullet \arrow[r, "\cong"] & \underline{\underline{H}}^i\tau\tau'\underline{\underline{A}}^\bullet \end{tikzcd} .$$

so $\underline{\underline{H}}^i\tau\underline{\underline{A}}^\bullet \to \underline{\underline{H}}^i\tau\tau'\underline{\underline{A}}^\bullet$ is an isomorphism. Then $\tau\underline{\underline{A}}^\bullet \to \tau\tau'\underline{\underline{A}}^\bullet$ is a q.i. We also have a map $\tau'\underline{\underline{A}}^\bullet \to \tau(\tau'\underline{\underline{A}}^\bullet)$ which induces isomorphisms on stalk cohomology for degrees $\geq p$, and $\underline{\underline{H}}^i\tau'\underline{\underline{A}}^\bullet = H^i\tau\tau'\underline{\underline{A}}^\bullet = 0$ for $i < p$, so this is a q.i. We then have a q.i. $\tau\underline{\underline{A}}^\bullet \xrightarrow{qi} \tau\tau'\underline{\underline{A}}^\bullet \xleftarrow{qi} \tau'\underline{\underline{A}}^\bullet$ in $D^b(X)$, and it's clear that these maps commute with induced maps $\tau\underline{\underline{A}}^\bullet \to \tau\underline{\underline{B}}^\bullet$, $\tau'\underline{\underline{A}}^\bullet \to \tau'\underline{\underline{B}}^\bullet$.

The proof of the other statement is similar.

We can easily define functors $\tau_{\leq p}$ and $\tau^{\geq p}$ satisfying the above; let $\tau_{\leq p}(\underline{\underline{A}}^{\bullet}) = \cdots \to \underline{\underline{A}}^{p-2} \xrightarrow{\partial^{p-2}} \underline{\underline{A}}^{p-1} \xrightarrow{\partial^{p-1}} \underline{\underline{\mathrm{Ker}}}\,\partial^{p} \to 0 \to 0 \to \cdots$

and $\qquad \tau^{\geq p}(\underline{\underline{A}}^{\bullet}) = \cdots \to 0 \to 0 \to \underline{\underline{\mathrm{Cok}}}\,\partial^{p-1} \xrightarrow{\partial^{p}} \underline{\underline{A}}^{p+1} \xrightarrow{\partial^{p+1}} \underline{\underline{A}}^{p+2} \to \cdots$.

There are canonical maps $\tau_{\leq p}\underline{\underline{A}}^{\bullet} \to \underline{\underline{A}}^{\bullet}$ and $\underline{\underline{A}}^{\bullet} \to \tau^{\geq p}\underline{\underline{A}}^{\bullet}$ and given a chain map $\underline{\underline{A}}^{\bullet} \to \underline{\underline{B}}^{\bullet}$ there is a natural way to define maps $\tau_{\leq p}\underline{\underline{A}}^{\bullet} \to \tau_{\leq p}\underline{\underline{B}}^{\bullet}$ and $\tau^{\geq p}\underline{\underline{A}}^{\bullet} \to \tau^{\geq p}\underline{\underline{B}}^{\bullet}$. Maps homotopic to zero go to maps homotopic to zero, so $\tau_{\leq p}$ and $\tau^{\geq p}$ are defined on $K^{b}(X)$. Finally, $\tau_{\leq p}$ and $\tau^{\geq p}$ take q.i.'s to q.i.'s so these functors make sense in $D^{b}(X)$.

We could also define the functors as follows:

$$\tilde{\tau}_{\leq p}(\underline{\underline{A}}^{\bullet}) = \cdots \to \underline{\underline{A}}^{p-1} \xrightarrow{\partial^{p-1}} \underline{\underline{A}}^{p} \xrightarrow{\partial^{p}} \underline{\underline{\mathrm{Im}}}\,\partial^{p} \to 0 \to 0 \to \cdots$$

$$\tilde{\tau}^{\geq p}(\underline{\underline{A}}^{\bullet}) = \cdots \to 0 \to 0 \to \underline{\underline{\mathrm{Im}}}\,\partial^{p-1} \xrightarrow{\partial^{p-1}} \underline{\underline{A}}^{p} \xrightarrow{\partial^{p}} \underline{\underline{A}}^{p+1} \to \cdots$$

There are canonical maps $\tilde{\tau}_{\leq p}\underline{\underline{A}}^{\bullet} \to \underline{\underline{A}}^{\bullet}$ and $\underline{\underline{A}}^{\bullet} \to \tilde{\tau}^{\geq p}\underline{\underline{A}}$, and these functors extend naturally to $D^{b}(X)$. To see that $\tau_{\leq p}\underline{\underline{A}}^{\bullet} \xrightarrow{qi} \tilde{\tau}_{\leq p}\underline{\underline{A}}^{\bullet}$ and $\tau^{\geq p}\underline{\underline{A}}^{\bullet} \xrightarrow{qi} \tilde{\tau}^{\geq p}\underline{\underline{A}}^{\bullet}$, one can easily explicitly set up a chain map between them, or note that the fact follows from the above theorem.

CHAPTER THREE

DUALITY

§3.1. The Category $D_f^b(X)$

Let X be a cell complex, and let $S_f(X) \subseteq S(X)$ be the subcategory of sheaves $\underline{\underline{A}}$ with $\dim \underline{\underline{A}}(\sigma) < \infty \ \forall \sigma \in X$. We can then let $K_f^b(X) \subseteq K^b(X)$ be the category of bounded chain complexes of elements of $S_f(X)$, with morphisms being chain maps modulo chain maps homotopic to zero. If we replace $S(X)$ with $S_f(X)$ and $K^b(X)$ with $K_f^b(X)$, all theorems and proofs in Chapter 1 and §2.1 are still valid; in particular, the functors described preserve the finite-dimensionality of stalks, and injectives are sheaves isomorphic to $\bigoplus_{\sigma \in X} [\sigma]^{V_\sigma}$ with V_σ being finite-dimensional vector spaces (similarly for projective sheaves).

The constructions of §2.2 can be made with $S_f(X)$, resulting in the derived category $D_f^b(X)$. Hence the objects of $D_f^b(X)$ are those of $K_f^b(X)$ and morphisms are represented by diagrams $\underline{\underline{A}}^\bullet \rightarrow \underline{\underline{C}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet$ in $K_f^b(X)$ where diagrams are identified in the same manner as in $D^b(X)$. All the theorems and proofs in Chapter 2 are valid if $D^b(X)$ is replaced with $D_f^b(X)$.

There is a functor $i : D_f^b(X) \rightarrow D^b(X)$ which takes an object to itself and the morphism represented by $\underline{\underline{A}}^\bullet \rightarrow \underline{\underline{C}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet$ in $D_f^b(X)$ to the morphism in $D^b(X)$ represented by the same diagram.

<u>Theorem 3.1.1:</u> The functor i is fully faithful.

<u>Proof:</u> We want to show that $\mathrm{Hom}_{D_f^b(X)}(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet) \to \mathrm{Hom}_{D^b(X)}(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)$ is bijective. Given $\tilde{f}$ in $D^b(X)$ represented by $\underline{\underline{A}}^\bullet \to \underline{\underline{C}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet$, let $\underline{\underline{B}}^\bullet \to \underline{\underline{I}}^\bullet$ be an injective resolution in $D_f^b(X)$. We can then construct a diagram

$$\begin{tikzcd}
& \underline{\underline{C}}^\bullet \arrow[dd, "qi"] & \\
\underline{\underline{A}}^\bullet \arrow[ur] \arrow[dr] & & \underline{\underline{B}}^\bullet \arrow[ul, "qi"'] \arrow[dl, "qi"] \\
& \underline{\underline{I}}^\bullet &
\end{tikzcd}$$

so $\underline{\underline{A}}^\bullet \to \underline{\underline{I}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet$ is a morphism in $D_f^b(X)$ representing $\tilde{f}$. We can use a similar argument to show that if two diagrams $\underline{\underline{A}}^\bullet \to \underline{\underline{C}}^\bullet \xleftarrow{qi} \underline{\underline{B}}^\bullet$ in $D_f^b(X)$ are identified in $D^b(X)$ then they are identified in $D_f^b(X)$.

Theorem 3.1.1 shows that we can consider $D_f^b(X)$ as a full subcategory of $D^b(X)$.

<u>Lemma 3.1.2:</u> Let $V^\bullet$ be a chain complex of vector spaces with $\dim H^i(V^\bullet) < \infty\ \forall i$ and for each i let $A^i \subseteq V^i$ be a finite-dimensional subspace. Then there is a subcomplex $W^\bullet \subseteq V^\bullet$ with $A^i \subseteq W^i$ and $\dim W^i < \infty\ \forall i$, such that $W^\bullet \hookrightarrow V^\bullet$ is a quasi-isomorphism.

Proof: We can find subspaces $H^i \subseteq \ker(\partial^i)$ such that $H^\bullet \hookrightarrow V^\bullet$ is a quasi-isomorphism, where the boundary maps of $H^\bullet$ are 0. Then $\dim H^i = \dim H^i(V^\bullet) < \infty$, so by replacing A^i with $A^i + H^i$, we can assume that $H^i \subseteq A^i$. We can further assume that $A^\bullet$ is a subcomplex of $V^\bullet$ by replacing A^i with $A^i + d^{i-1}(A^{i-1})$. Find $T^i \subseteq A^i$ for each i where $A^i \cap \partial^{i-1}(V^{i-1}) = \partial^{i-1}(A^{i-1}) \oplus T^i$, and then find $S^{i-1} \subseteq V^{i-1}$ where $\partial^{i-1}(S^{i-1}) = T^i$ and $\partial^{i-1}|_{S^{i-1}}$ is one-to-one. $A^\bullet + S^\bullet$ is then a subcomplex of $V^\bullet$ with $\dim(A^i + S^i) < \infty$. The composition $H^\bullet \hookrightarrow A^\bullet + S^\bullet \hookrightarrow V^\bullet$ shows that $H^i(A^\bullet + S^\bullet) \to H^i(V^\bullet)$ is surjective. To see that this map is injective, let $a+s \in A^i + S^i$ be in $\operatorname{Im}(\partial^{i-1})$. Then $\partial^i(a+s) = 0$ and since $\partial^i(a) \in \partial^i(A^i)$, $\partial^i(s) \in T^{i+1}$, and $\partial^i(A^i)$ and T^{i+1} are independent, we have $\partial^i(a) = \partial^i(s) = 0$, so $s = 0$. Hence $a+s = a \in A^i \cap \partial^{i-1}(V^{i-1}) = \partial^{i-1}(A^{i-1}) \oplus \partial^{i-1}(S^{i-1})$, so $a \in \partial^{i-1}(A^{i-1} + S^{i-1})$. Then $H^i(A^\bullet + S^\bullet) \to H^i(V^\bullet)$ is injective, so $A^\bullet + S^\bullet \hookrightarrow V^\bullet$ is a quasi-isomorphism.

Theorem 3.1.3: Let $D^b_{fc}(X)$ be the full subcategory of $D^b(X)$ consisting of objects $\underline{\underline{A}}^\bullet$ with $\dim \underline{\underline{H}}^i(\underline{\underline{A}}^\bullet)(\sigma) < \infty$ for all $i \in \mathbb{Z}$ and $\sigma \in X$. Then the inclusion $D^b_f(X) \hookrightarrow D^b_{fc}(X)$ is an equivalence of categories.

Proof: This just says that every element of $D^b_{fc}(X)$ is isomorphic to an element of $D^b_f(X)$. Let $\underline{\underline{A}}^\bullet \in D^b_{fc}(X)$. We will construct a subcomplex of sheaves $\underline{\underline{S}}^\bullet \subseteq \underline{\underline{A}}^\bullet$ in $D^b_f(X)$ with $\underline{\underline{S}}^\bullet \hookrightarrow \underline{\underline{A}}^\bullet$ a quasi-isomorphism.

We can construct $\underline{\underline{S}}^\bullet$ over the 0-skeleton of X by choosing a subcomplex $\underline{\underline{S}}^\bullet(\sigma) \subseteq \underline{\underline{A}}^\bullet(\sigma)$ with $\dim \underline{\underline{S}}^i(\sigma) < \infty$ and $\underline{\underline{S}}^\bullet(\sigma) \hookrightarrow \underline{\underline{A}}^\bullet(\sigma)$ a quasi-isomorphism for each 0-cell σ. Suppose $\underline{\underline{S}}^\bullet$ has been constructed over the (k-1)-skeleton. For τ a k-dimensional cell, let $R^i_\tau \subseteq \underline{\underline{A}}^i(\tau)$ be the subspace generated by $\bigcup_{\gamma\in\tau} p^{\underline{\underline{A}}^i}_{\gamma,\tau}(\underline{\underline{S}}^i(\gamma))$. Then R^i_τ is finite dimensional, so by lemma 3.1.2, there is a subcomplex $\underline{\underline{S}}^\bullet(\tau) \subseteq \underline{\underline{A}}^\bullet(\tau)$ containing the R^{i}_τ's where $\dim \underline{\underline{S}}^i(\tau) < \infty$ and $\underline{\underline{S}}^\bullet(\tau) \hookrightarrow \underline{\underline{A}}^\bullet(\tau)$ is a quasi-isomorphism. Since the corestriction map $p^{\underline{\underline{A}}^i}_{\gamma,\tau}$ maps $\underline{\underline{S}}^i(\gamma)$ into $\underline{\underline{S}}^i(\tau)$, doing this over each k-cell extends $\underline{\underline{S}}^\bullet$ to the k-skeleton. By induction, this constructs all of $\underline{\underline{S}}^\bullet$.

§3.2. Biduality

Definition: Let X be a cell complex. Then the dualizing complex $\underline{\underline{\mathbb{D}}}^\bullet_X \in D^b(X)$ has in degree $-i$ the sheaf $\underline{\underline{\mathbb{D}}}^{-i}_X = \bigoplus_{\sigma^i\in X} [\sigma]$. The boundary map $\underline{\underline{\mathbb{D}}}^{-i}_X \to \underline{\underline{\mathbb{D}}}^{-i+1}_X$ is the zero map between components $[\sigma]$ and $[\tau]$ if $\tau \nless \sigma$ and is given by multiplication by $[\sigma:\tau]$ if $\tau < \sigma$.

Proposition 3.2.1: Different choices of orientations for the cells of X lead to the same definition of $\underline{\underline{\mathbb{D}}}^\bullet_X$, up to isomorphism.

Proof: Let $\underline{\underline{\mathbb{D}}}^\bullet_X$ and $\underline{\underline{\mathbb{D}}}^{\bullet\prime}_X$ be the dualizing complexes for two different choices of orientations. Then there is a chain isomorphism $\underline{\underline{\mathbb{D}}}^\bullet_X \to \underline{\underline{\mathbb{D}}}^{\bullet\prime}_X$ which maps $[\tau] \to [\tau]$ by $\pm\,\mathrm{id}$, the sign depending on whether the orientation choices are the same or different on τ.

Theorem 3.2.2: (i) $\mathbb{H}^{-i}(X,\underline{\underline{\mathbb{D}}}^\bullet_X) = H^{CS}_i(|X|,\mathbb{Q})$ = homology with closed supports or Borel-Moore homology.

(ii) $\mathbb{H}_c^{-i}(X,\underline{\underline{\mathbb{D}}}_X^\bullet) = H_i(|X|,\mathbb{Q})$ = ordinary homology.

Proof: (i) follows from the fact that $\underline{\underline{\mathbb{D}}}_X^\bullet$ is an injective complex, so $\mathbb{H}^{-i}(X,\underline{\underline{\mathbb{D}}}_X^\bullet) \cong H^{-i}(\Gamma^\bullet(X,\underline{\underline{\mathbb{D}}}_X^\bullet))$, and $\Gamma^{-i}(X,\underline{\underline{\mathbb{D}}}_X^\bullet)$ is exactly the group of i-dimensional regular CW chains with closed support on $|X|$.

If X is compact, (ii) is clear by the same reasoning as in (i). Otherwise, let $X \cup \{p\} = \overline{X}$ be the one-point compactification of X. Then $|\overline{X} - st(p)|$ is a deformation retract of $|X|$, so $H_i(|X|,\mathbb{Q}) \cong H_i(|\overline{X} - st(p)|,\mathbb{Q}) \cong H^{-i}(\Gamma_c^\bullet(X,\underline{\underline{\mathbb{D}}}_X^\bullet)) \cong \mathbb{H}_c^i(X,\underline{\underline{\mathbb{D}}}_X^\bullet)$.

Lemma 3.2.3: Let $X = \overline{\sigma}$ be a compact cell complex consisting of a cell σ and its faces. Let $\gamma < \sigma$ be a proper face and let $S_\gamma^\bullet \subseteq C^\bullet(X,\underline{\mathbb{Q}})$ be the subcomplex

$$\cdots \to \bigoplus_{\tau^k \in st(\gamma)} \mathbb{Q} \longrightarrow \bigoplus_{\tau^{k+1} \in st(\gamma)} \mathbb{Q} \to \cdots .$$

Then $S_\gamma^\bullet$ is exact.

Proof: We have a short exact sequence of chain complexes

$$0 \to S_\gamma^\bullet \to C^\bullet(X,\underline{\mathbb{Q}}) \to C^\bullet(X-st(\gamma),\underline{\mathbb{Q}}) \to 0 .$$

$H^i(X,\underline{\mathbb{Q}})$ is the CW cohomology of $|X|$, hence is 0 for $i > 0$, and

$H^i(X\text{-st}(\gamma),\underline{\underline{Q}})$ is the CW cohomology of $|X - st(\gamma)|$ which is the deformation retract of $|X|$ minus a point on its boundary, so it is also 0 for $i > 0$. Since the map $\Gamma(X,\underline{\underline{Q}}) \to \Gamma(X\text{-st}(\gamma),\underline{\underline{Q}})$ by restriction of sections is an isomorphism, we have that $H^0(X,\underline{\underline{Q}}) \to H^0(X\text{-st}(\gamma),\underline{\underline{Q}})$ is an isomorphism. The lemma now follows from the long exact sequence associated to the above short exact sequence.

<u>Definition</u>: Given $\underline{\underline{A}}^{\bullet} \in D^b(X)$, the <u>dual</u> of $\underline{\underline{A}}^{\bullet}$ is defined to be $D\underline{\underline{A}}^{\bullet} = R^{\bullet}\underline{\underline{Hom}}(\underline{\underline{A}}^{\bullet},\underline{\underline{D}}_X^{\bullet}) = \underline{\underline{Hom}}^{\bullet}(\underline{\underline{A}}^{\bullet},\underline{\underline{D}}_X^{\bullet})$ (since $\underline{\underline{D}}_X^{\bullet}$ is injective). D is then a contravariant functor $D : D^b(X) \to D^b(X)$.

<u>Theorem 3.2.4</u>(Biduality): For $\underline{\underline{A}}^{\bullet} \in D_f^b(X)$, we have $\underline{\underline{A}}^{\bullet} \overset{qi}{\cong} DD\underline{\underline{A}}^{\bullet}$.

<u>Proof</u>: It suffices to show the result for an injective complex $\underline{\underline{I}}^{\bullet} \in D_f^b(X)$.

We begin with the case of a single elementary injective $[\sigma]^V$ with $\dim \sigma = r$. $\underline{\underline{Hom}}^{\bullet}([\sigma]^V,\underline{\underline{D}}_X^{\bullet})$ has in degree $-k$, the sheaf $\underline{\underline{Hom}}([\sigma]^V, \bigoplus_{\tau^k \in X} [\tau^k]) = \bigoplus_{\tau^k \leq \sigma} [\tau^k]^{V*}$, and the component maps $[\tau]^{V*} \to [\gamma]^{V*}$ for $\gamma <_1 \tau \leq \sigma$ are given by multiplication by $[\tau:\gamma]$. The dual of this is given by a double complex which has in position $(n,-m)$, the sheaf $\underline{\underline{Hom}}(\bigoplus_{\tau^n \leq \sigma} [\tau^n]^{V^*}, \bigoplus_{\gamma^m \in X} [\gamma^m]) = \bigoplus_{\gamma^m \leq \tau^n \leq \sigma} [\gamma^m]^{V^{**}}_{\tau^n}$ (the subscript τ^n is being used only to distinguish different copies of $[\gamma]^{V^{**}}$). It's easily seen that all component maps in the double complex are given by multiplication by orientation: given $\lambda^{m-1} < \gamma^m \leq \tau^n \leq \sigma$, we have component map

$[\gamma^m]V^{**}_{\tau^n} \xrightarrow{[\gamma^m:\lambda^{m-1}]} [\lambda^{m-1}]V^{**}_{\tau^n}$ and given $\gamma^m \le \tau^n < \mu^{n+1} \le \sigma$, we have component map $[\gamma^m]V^{**}_{\tau^n} \xrightarrow{[\mu^{n+1}:\tau^n]} [\gamma^m]V^{**}_{\mu^{n+1}}$. Hence $DD[\sigma]^V$ is given by the following (the ordered pair gives the position in the double complex):

$$\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
 & & & & 0 \arrow[r] \arrow[d] & \bigoplus\limits^{(0,0)}_{\gamma^0=\tau^0\le\sigma}[\gamma^0]^{V^{**}}_{\tau^0} \arrow[r] \arrow[d] & 0 \\
 & & & 0 \arrow[r] & \bigoplus\limits^{(1,-1)}_{\gamma^1=\tau^1\le\sigma}[\gamma^1]^{V^{**}}_{\tau^1} \arrow[r] \arrow[d] & \bigoplus\limits^{(1,0)}_{\gamma^0<\tau^1\le\sigma}[\gamma^0]^{V^{**}}_{\tau^1} \arrow[r] \arrow[d] & 0 \\
 & & & & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & & 0 \arrow[d] & & & & \\
 & 0 \arrow[r] \arrow[d] & \bigoplus\limits^{(r-1,-r+1)}_{\gamma^{r-1}=\tau^{r-1}\le\sigma}[\gamma^{r-1}]^{V^{**}}_{\tau^{r-1}} \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus\limits^{(r-1,-1)}_{\gamma^1<\tau^{r-1}\le\sigma}[\gamma^1]^{V^{**}}_{\tau^{r-1}} \arrow[r] \arrow[d] & \bigoplus\limits^{(r-1,0)}_{\gamma^0<\tau^{r-1}\le\sigma}[\gamma^0]^{V^{**}}_{\tau^{r-1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {[\sigma]^{V^{**}}_{\sigma}}^{(r,-r)} \arrow[r] \arrow[d] & \bigoplus\limits^{(r,-r+1)}_{\gamma^{r-1}<\tau^r=\sigma}[\gamma^{r-1}]^{V^{**}}_{\tau^r} \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus\limits^{(r,-1)}_{\gamma^1<\tau^r=\sigma}[\gamma^1]^{V^{**}}_{\tau^r} \arrow[r] \arrow[d] & \bigoplus\limits^{(r,0)}_{\gamma^0<\tau^r=\sigma}[\gamma^0]^{V^{**}}_{\tau^r} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 & 0 &
\end{tikzcd}$$

It follows from lemma 3.2.3 that every column except the $-r^{th}$ is exact. If we think of $[\sigma]^{V^{**}}$ as a double complex with 0 in all but the $(-r,r)^{th}$ position, then we have a map $\pi^{\bullet\bullet} : DD[\sigma]^V \to [\sigma]^{V^{**}}$ of double complexes that gives a quasi-isomorphism between columns. Then this map is a quasi-isomorphism between associated single complexes, and hence $[\sigma]^{V^{**}} \underset{=}{\underline{qi}} DD[\sigma]^V$.

We will now consider $[\sigma]^{V^{**}}$ and $DD[\sigma]^V$ as single complexes, where the sign convention used to form a single complex from a double complex is $\{\mu_k^{ij}\}$, $k=1,2$, where $\mu_2^{ij} = +1 \; \forall\, i,j$ and $\mu_1^{ij} = (-1)^{n+m+1}$. We define a map $i_\sigma : [\sigma]^{V^{**}} \to DD[\sigma]^V$ consisting, in degree 0, of component maps $[\sigma]^{V^{**}} \to [\gamma]_\gamma^{V^{**}}$ given by the identity map for each $\gamma \leq \sigma$. To check that this is a chain map, we must show that the composition $[\sigma]^{V^{**}} \to (DD[\sigma]^V)^0 \to (DD[\sigma]^V)^1$ is 0. Given a summand $[\gamma]_\tau^{V^{**}}$ of $(DD[\sigma]^V)^1$ (hence $\gamma <_1 \tau \leq \sigma$), the component map $[\sigma]^{V^{**}} \to [\gamma]_\tau^{V^{**}}$ is the sum of two maps:

$$\begin{tikzcd}
 & {[\gamma]_\gamma^{V^{**}}} \arrow[dr, "-"] & \\
{[\sigma]^{V^{**}}} \arrow[ur, "+"] \arrow[dr, "+"'] & & {[\gamma]_\tau^{V^{**}}} \\
 & {[\tau]_\tau^{V^{**}}} \arrow[ur, "+"'] &
\end{tikzcd}$$

Each map in the diagram is given by multiplication by ± 1 as indicated. But the sum of these is 0, so i_σ is a chain map.

Note that the composition $[\sigma]^{V^{**}} \xrightarrow{i_\sigma} DD[\sigma]^V \xrightarrow{\pi^\bullet} [\sigma]^{V^{**}}$ ($\pi^\bullet$ induced from $\pi^{\bullet\bullet}$) is the identity and $\pi^\bullet$ is a quasi-isomorphism, hence i_σ is a quasi-isomorphism.

Now let $\alpha : [\sigma]^V \to [\lambda]^W$ be a map given by $a : V \to W$. This induces $\tilde{\alpha} : DD[\sigma]^V \to DD[\lambda]^W$ which, for $\mu^n \le \tau^m \le \lambda \le \sigma$, has component map $[\mu^n]^{V^{**}}_{\tau^m} \to [\mu^n]^{W^{**}}_{\tau^m}$ in position $(-n,m)$ given by a^{**} (all other component maps are 0). It's clear that we have a commutative diagram

$$\begin{tikzcd}
{[\sigma]^{V^{**}}} \arrow[r, "\alpha^{**}"] \arrow[d, "i_\sigma"] & {[\lambda]^{W^{**}}} \arrow[d, "i_\lambda"] \\
DD[\sigma]^V \arrow[r, "\tilde{\alpha}"] & DD[\lambda]^W
\end{tikzcd}$$

where α^{**} is given by a^{**}.

If $\underline{\underline{I}}^\bullet$ is an injective complex, then $DD\underline{\underline{I}}^\bullet$ is given by a triple complex, or a single complex of double complexes of the form $\bigoplus_{\sigma\in X} DD[\sigma]^{V_\sigma}$. Then by using the quasi-isomorphisms i_σ constructed above, we can define a map $i : \underline{\underline{I}}^{\bullet **} \to DD\underline{\underline{I}}^\bullet$ where $\underline{\underline{I}}^{\bullet **}$ is the injective complex which replaces each summand $[\sigma]^V$ in $\underline{\underline{I}}^\bullet$ with $[\sigma]^{V^{**}}$, and each component map $\alpha : [\sigma]^V \to [\tau]^W$ with α^{**}. It follows from theorem 1.2.3 that i is an isomorphism. But since the $\underline{\underline{I}}^i$ have finite-dimensional stalks, the components $[\sigma]^{V^{**}}$ of $\underline{\underline{I}}^{\bullet **}$ are isomorphic to $[\sigma]^V$, so $\underline{\underline{I}}^\bullet \cong \underline{\underline{I}}^{\bullet **}$. The result now follows.

Remark: Given $\underline{\underline{A}}^\bullet \in D^b(X)$ we can define $\underline{\underline{A}}^{\bullet **} \in D^b(X)$ by letting $(\underline{\underline{A}}^{\bullet **})(\sigma) = (\underline{\underline{A}}^\bullet(\sigma))^{**}$. This construction is consistent with the construction of $\underline{\underline{I}}^{\bullet **}$ made in the above proof for $\underline{\underline{I}}^\bullet$ an injective complex. Since $V \longmapsto V^{**}$ is an exact functor on the category of vector spaces, given a quasi-isomorphism $\underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet$, the induced map $\underline{\underline{A}}^{\bullet **} \to \underline{\underline{B}}^{\bullet **}$ is a quasi-isomorphism. Then it follows from the above proof that for $\underline{\underline{A}}^\bullet \in D^b(X)$, we have $\underline{\underline{A}}^{\bullet **} \overset{qi}{=} DD\underline{\underline{A}}^\bullet$ since $\underline{\underline{A}}^{\bullet **} \overset{qi}{=} \underline{\underline{I}}^{\bullet **} \overset{qi}{=} DD\underline{\underline{I}}^\bullet \overset{qi}{=} DD\underline{\underline{A}}^\bullet$ for $\underline{\underline{A}}^\bullet \to \underline{\underline{I}}^\bullet$ an injective resolution.

§3.3. The Functors $R^\bullet f_!$ and $f^!$

In general, it will be necessary to assume further conditions on cellular maps $f : X \to Y$ when working with the functor $R^\bullet f_!$. In particular, these conditions will imply that $R^\bullet f_!$ corresponds to the functor $R^\bullet f_!$ used in the standard theory of derived categories.

Definition: A cellular map $f : X \to Y$ is called a fibred cellular map if:

(i) given $\sigma \in X$ with $f(\sigma) = \tau$, there is a homeomorphism $h : |C| \times |\tau| \to |f^{-1}(\tau) \cap \overline{\sigma}|$ commuting with f and the projection $|C| \times |\tau| \to |\tau|$, where C is a cell complex and the cells of $f^{-1}(\tau) \cap \overline{\sigma}$ are exactly the sets $h(|\gamma| \times |\tau|)$ for $\gamma \in C$, and

(ii) given $\sigma \in X$ and $y \in |\overline{f(\sigma)}|$, then $H^i_c(f^{-1}(y) \cap |\overline{\sigma}|, \mathbb{Q}) = 0$ for $i \neq 0$.

Remark: It follows immediately from theorem 3.3.1 below (which uses only condition (i) of the definition of a fibred map) that condition (ii) can be stated purely in terms of the cellular structures of X and Y:

(ii') Given $\sigma \in X$ and $\lambda^r \leq f(\sigma)$, the following chain complex is exact except in degree r, where the boundary maps are defined similarly to those in $C^\bullet_c(X,\underline{\mathbb{Q}})$.

$$\cdots \longrightarrow \overset{\text{deg.}k}{\bigoplus_{\substack{\gamma^k \leq \sigma \\ f(\gamma^k)=\lambda}}} \mathbb{Q} \longrightarrow \overset{\text{deg.}k+1}{\bigoplus_{\substack{\gamma^{k+1} \leq \sigma \\ f(\gamma^{k+1})=\lambda}}} \mathbb{Q} \longrightarrow \cdots$$

Theorem 3.3.1: Given a fibred cellular map $f : X \to Y$ and $\tau \in Y$, there is a homeomorphism $h : |C| \times |\tau| \to f^{-1}(|\tau|)$ commuting with $f : f^{-1}(|\tau|) \to |\tau|$ and the projection $|C| \times |\tau| \to |\tau|$, where C is a cell complex and the cells of $f^{-1}(|\tau|)$ are exactly the sets $h(|\gamma| \times |\tau|)$ for $\gamma \in C$.

Proof: We will use induction on the dimension of X. If $\dim X = \dim \tau$, then $f^{-1}(\tau)$ is a disjoint union of copies of τ, so the theorem follows easily.

Assuming the result for $\dim X = k$, let $\dim X = k+1$, and let

X' be a k-skeleton of X (note that the skeletons of a cell complex are cell complexes since if $C \subseteq X$ is closed, and $|X| \cup \{\infty\}$ is the one-point compactification of $|X|$, then $|C| \cup \{\infty\}$ is the one-point compactification of $|C|$). We then have a homeomorphism $h : |A| \times |\tau| \to |f^{-1}(\tau) \cap X'|$ as in the statement of the theorem, for a cell complex A. Let σ be a (k+1)-cell of X with $f(\sigma) = \tau$ and g a similar homeomorphism $g : |C| \times |\tau| \to |f^{-1}(\tau) \cap \bar{\sigma}|$. Then C is a cell complex with a cell $\gamma \in C$ such that $C = \bar{\gamma}$.

Let $\tilde{A} \subseteq A$ be the subset consisting of those cells $\mu \in A$ where the cell $h(|\mu| \times |\tau|)$ is a proper face of σ. If $\alpha \leq \lambda \in \tilde{A}$, then by continuity $h(|\alpha| \times |\tau|)$ is a proper face of σ, so $|\tilde{A}|$ is closed in $|A|$, hence $\tilde{A}$ is a subcomplex. We then have a homeomorphism

$$r = (g|_{|\partial C| \times |\tau|})^{-1} \circ (h|_{|\tilde{A}| \times |\tau|}) : |\tilde{A}| \times |\tau| \to |\partial C| \times |\tau|$$

commuting with projection maps $|\tilde{A}| \times |\tau| \to |\tau|$ and $|\partial C| \times |\tau| \to |\tau|$ where ∂C is the subcomplex $C - \{\gamma\}$ of C, and where the correspondence $s : \tilde{A} \to \partial C$ given by $|s(\mu)| \times \{y\} = r(|\mu| \times \{y\})$ is the same for any $y \in |\tau|$.

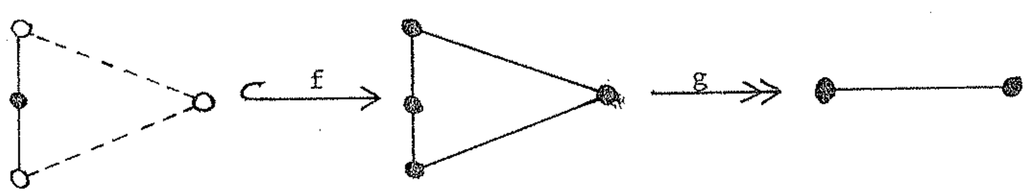
Suppose C is not compact. It's easily checked that r extends to a continuous map $\bar{r} : (|\tilde{A}| \cup \{\infty\}) \times |\tau| \to (|\partial C| \cup \{\infty\}) \times |\tau|$ where $|\tilde{A}| \cup \{\infty\}$ and $|\partial C| \cup \{\infty\}$ are one-point compactifications, by letting

$\bar{r}(\infty,y) = (\infty,y)$. $|\tilde{A}| \cup \{\infty\} \cong |\partial C| \cup \{\infty\}$ is homeomorphic to a sphere of dimension $\dim\sigma - \dim\tau - 1$, so we can attach a cell λ of dimension $\dim\sigma - \dim\tau$ to $\tilde{A} \cup \{\infty\}$, resulting in a compact cell complex $\tilde{B} \cup \{\infty\} = \bar{\lambda}$. We can always extend a homeomorphism $S^n \times \mathbb{R}^k \to S^n \times \mathbb{R}^k$ that commutes with the projection $S \times \mathbb{R}^k \to \mathbb{R}^k$ to a homeomorphism $B^{n+1} \times \mathbb{R}^k \to B^{n+1} \times \mathbb{R}^k$ commuting with the projection $B^{n+1} \times \mathbb{R}^k \to \mathbb{R}^k$ by extending each $S^n \times \{y\} \to S^n \times \{y\}$ along radial lines. Hence we have an extension of $\bar{r}$ to $|\tilde{B} \cup \{\infty\}| \times |\tau| \xrightarrow{\cong} |C \cup \{\infty\}| \times |\tau|$ which restricts to $r' : |\tilde{B}| \times |\tau| \xrightarrow{\cong} |C| \times |\tau|$, r' an extension of r . If C is compact, by the same kind of reasoning we can again find an extension $r' : |\tilde{B}| \times |\tau| \xrightarrow{\cong} |C| \times |\tau|$ of r where $\tilde{B}$ is the complex $\tilde{A}$ with a $(\dim\sigma - \dim\tau)$-cell attached.

A and $\tilde{B}$ together form a cell complex $B = A \cup \tilde{B}$ where $A \cap \tilde{B} = \tilde{A}$ (this will be a cell complex since $|A|$ and $|\tilde{B}|$ are closed in $|B|$, hence if $|B| \cup \{\infty\}$ is the one-point compactification of $|B|$, then $|A| \cup \{\infty\}$ and $|\tilde{B}| \cup \{\infty\} \subseteq |B| \cup \{\infty\}$ are also one-point compactifications). Note that on $|\tilde{A}| \times |\tau|$ the homeomorphism $g \circ r'$ is $g \circ g^{-1} \circ h = h$, so h and $g \circ r'$ combine to form a homeomorphism $h' : |B| \times |\tau| \longrightarrow |f^{-1}(\tau) \cap (X' \cup \{\sigma\})|$ commuting with f and the projection map $|B| \times |\tau| \to |\tau|$, where the cells of $f^{-1}(\tau) \cap (X' \cup \{\sigma\})$ are exactly the sets $h'(|\gamma| \times |\tau|)$ for $\gamma \in B$.

Applying this process to each $(k+1)$-cell in X shows that the theorem holds for X . Then the theorem follows by induction.

Remark: If $f : X \to Y$ and $g : Y \to Z$ are fibred cellular maps, it does not necessarily follow that gf is a fibred map. For example, let Y be the closed 2-simplex with an extra 0-cell put on one of its 1-dimensional faces σ, and let $X = Y - W$ where W is the union of the closures of the other two 1-dimensional faces. Let Z be the closed 1-simplex, and let $g : Y \to Z$ be the simplicial map of the 2-simplex to the 1-simplex which sends the vertices of σ to one vertex of Z and the third vertex of Y to the other vertex of Z. Let $f : X \to Y$ be the inclusion. Then f and g are fibred maps, but gf is not, since the fibres do not have the structure of a cell complex.



It is not clear whether or not the composition of two cellular maps is cellular.

For fibred maps, we have the following basic relation between $Rf_!$ and Rf_*.

Theorem 3.3.2: $R^\bullet f_! \underline{\underline{A}}^\bullet \overset{qi}{=} DR^\bullet f_* D\underline{\underline{A}}^\bullet$ for $\underline{\underline{A}}^\bullet \in D^b_f(X)$ and $f : X \to Y$ a fibred cellular map.

Proof: We will show that $f_! D\underline{\underline{I}}^\bullet \overset{qi}{=} Df_* \underline{\underline{I}}^\bullet$ for $\underline{\underline{I}}^\bullet$ an injective complex. The result then follows by taking an injective resolution of $\underline{\underline{A}}^{\bullet}$, and by theorem 3.2.4.

Given $\gamma \in Y$, let $h : |C| \times |\gamma| \to f^{-1}(|\gamma|)$ be a homeomorphism as in theorem 3.3.1. The map h shows that if $f(\tau) = \gamma$ and $\dim \tau = \dim \gamma + 1$, then there are either 1 or 2 cells $\mu <_1 \tau$ with $f(\mu) = \gamma$. If orientations are chosen on the cells of C , then by using h , orientations can be chosen on the cells of $f^{-1}(\gamma)$ so that $[\alpha:\beta] = [h(|\alpha| \times |\gamma|):h(|\beta| \times |\gamma|)]$ for $\beta \leq \alpha \in C$, and $f|_{|\tau|}$ is orientation preserving if $\dim \tau = \dim \gamma$, $f(\tau) = \gamma$. We will assume that for every $\gamma \in Y$, orientations on $f^{-1}(\gamma)$ are chosen as above for some homeomorphism $h : |C| \times |\gamma| \to f^{-1}(|\gamma|)$.

For $\lambda \in X$ and $\gamma \leq f(\lambda)$, γ a k-cell, let $C^i_\gamma(\lambda)$ $= \bigoplus_{\mu^{i+k} \in f^{-1}(\gamma) \cap \bar{\lambda}} \mathbb{Q}$. Then $C^{\bullet}_\gamma(\lambda)$ is the chain complex in condition (ii') in the remark after the definition of a fibred map, where $C^i_\gamma(\lambda)$ is in degree $i + k$. But this chain complex is clearly isomorphic to $C^{\bullet}_c(\tilde{C},\underline{\underline{\mathbb{Q}}})$ (with $C^i_\gamma(\lambda)$ corresponding to $C^i_c(\tilde{C},\underline{\underline{\mathbb{Q}}})$), where $\tilde{C}$ are the cells of C corresponding to cells in $\bar{\lambda}$ via h so $\ker(C^0_\gamma(\lambda) \to C^1_\gamma(\lambda)) = \ker(C^0_c(\tilde{C},\underline{\underline{\mathbb{Q}}}) \to C^1_c(\tilde{C},\underline{\underline{\mathbb{Q}}})) = \Gamma_c(\tilde{C},\underline{\underline{\mathbb{Q}}}) =$ $\{\alpha \in \Gamma(f^{-1}(\gamma) \cap \bar{\lambda},\underline{\underline{\mathbb{Q}}}) | \alpha(\mu) = 0$ if $f^{-1}(y) \cap |\bar{\mu}|$ is not compact for $y \in |\gamma|\} = f_![\lambda](\gamma)$. We then have an exact sequence

$$0 \to f_![\lambda](\gamma) \to C^0_\gamma(\lambda) \to C^1_\gamma(\lambda) \to \cdots$$

Let $\sigma \in X$ with $\dim \sigma = n$ and $\dim f(\sigma) = k$. Then $f_! D[\sigma]$ is the chain complex

$$f_![\sigma] \longrightarrow \bigoplus_{\tau^{n-1}<\sigma} f_![\tau^{n-1}] \longrightarrow \bigoplus_{\tau^{n-2}<\sigma} f_![\tau^{n-2}] \rightarrow \cdots \rightarrow \bigoplus_{\tau^0<\sigma} f_![\tau^0]$$

beginning in degree $-n$ where the components of the boundary maps are given by multiplication by $[\tau^i:\tau^{i-1}]$. The chain complex $Df_*[\sigma]$ is

$$[f(\sigma)] \longrightarrow \bigoplus_{\gamma^{k-1}<f(\sigma)} [\gamma^{k-1}] \longrightarrow \bigoplus_{\gamma^{k-2}<f(\sigma)} [\gamma^{k-2}] \rightarrow \cdots \rightarrow \bigoplus_{\gamma^0<f(\sigma)} [\gamma^0]$$

beginning in degree $-k$, boundary maps having components given by multiplication by $[\gamma^i:\gamma^{i-1}]$. Given $\tau \in X$ with $\dim \tau = \dim f(\tau)$, there is a unique map $f_![\tau] \to [f(\tau)]$ which is $\mathrm{id} : \mathbb{Q} \to \mathbb{Q}$ over $f(\tau)$. We then can define a map $f_! D[\sigma] \to Df_*[\sigma]$ which in degree $-i$ is 0 on $f_![\tau]$ for $\dim \tau > \dim f(\tau)$ and is the above map $f_![\tau] \to [f(\tau)]$ for $\dim \tau = \dim f(\tau)$. We claim that this is a chain map.

We want to show that the following commutes.

$$\begin{array}{ccc}
\bigoplus_{\tau^{i+1}\le\sigma} f_![\tau^{i+1}] & \longrightarrow & \bigoplus_{\gamma^{i+1}\le f(\sigma)} [\gamma^{i+1}] \\
\downarrow & & \downarrow \\
\bigoplus_{\tau^{i}\le\sigma} f_![\tau^{i}] & \longrightarrow & \bigoplus_{\gamma^{i}\le f(\sigma)} [\gamma^{i}]
\end{array} \quad .$$

Given $\tau^{i+1} \le \sigma$ and $\gamma^i \le f(\sigma)$, both compositions will give the 0-map for the component $f_![\tau^{i+1}] \to [\gamma^i]$, if $\dim f(\tau^{i+1}) \le i - 1$. Suppose $\dim f(\tau^{i+1}) = i$. Then the top map is zero. There are either 1 or 2 cells $\mu <_1 \tau^{i+1}$ with $f(\mu) = f(\tau^{i+1})$, i.e., cells $\mu <_1 \tau^{i+1}$ with $\dim \mu = \dim f(\mu)$. Suppose first there is a unique such μ . Then for $y \in |f(\tau^{i+1})|$, $f^{-1}(y) \cap |\overline{\tau^{i+1}}|$ is a 1-cell with only one 0-cell as a face, so it is not compact. Hence $f_![\tau^{i+1}](f(\tau^{i+1})) = 0$, so the composition $f_![\tau^{i+1}] \to f_![\mu] \to [f(\tau^{i+1})]$ is 0 over $f(\tau^{i+1})$, which implies that the composition is 0 . In this case, then, the components $f_![\tau^{i+1}] \to [f(\tau^{i+1})]$ agree. Now suppose there are two $\mu_j <_1 \tau^{i+1}$ with $f(\mu_j) = f(\tau^{i+1})$, $j = 1,2$. Then we have the composition

$$\begin{tikzcd}[row sep=small] & f_![\mu_1] \arrow[dr] & \\ f_![\tau^{i+1}] \arrow[ur] \arrow[dr] & \oplus & {[f(\tau^{i+1})]} \\ & f_![\mu_2] \arrow[ur] & \end{tikzcd}$$

But the maps $f_![\tau^{i+1}] \to f_![\mu_j]$ over $f(\tau^{i+1})$ are negatives of each other since $[\tau^{i+1}:\mu_1] = -[\tau^{i+1}:\mu_2]$, so this composition is zero over $f(\tau^{i+1})$, hence the map is zero. The components $f_![\tau^{i+1}] \to [f(\tau^{i+1})]$ agree in this case too, then.

This leaves the case $\dim f(\tau^{i+1}) = i+1$. Both compositions are zero on $f_![\tau^{i+1}] \to [\gamma^i]$ if $\gamma^i \nless f(\tau^{i+1})$. If $\gamma^i < f(\tau^{i+1})$ but there is no $\mu \le \tau^{i+1}$ with $f(\mu) = \gamma^i$, then $f_![\tau^{i+1}]$ is 0 over $\overline{\gamma^i}$, so the only possible map $f_![\tau^{i+1}] \to [\gamma^i]$ is the 0-map. Suppose we have $\mu^i \le \tau^{i+1}$, $\gamma^i \le f(\tau^{i+1})$, $f(\mu^i) = \gamma^i$. Since f is orientation preserving on μ^i and τ^{i+1} , the maps

$$\begin{tikzcd} f_![\tau^{i+1}] \arrow[r] \arrow[d] & {[f(\tau^{i+1})]} \arrow[d] \\ f_![\mu^i] \arrow[r] & {[\gamma^i]} \end{tikzcd}$$

commute. Finally, there can be no more than one $\mu \leq \tau^{i+1}$ with $f(\mu) = \gamma^i$ since for $y \in |\gamma^i|$, $f^{-1}(y) \cap |\overline{\tau^{i+1}}|$ is a finite set of points, hence compact, so by lemma 1.6.1, it is connected; i.e., it is a single point. This completes the proof of the claim.

Note that given a map $[\sigma] \to [\tau]$, the induced maps

$$\begin{array}{ccc} f_! D[\tau] & \longrightarrow & Df_*[\tau] \\ \downarrow & & \downarrow \\ f_! D[\sigma] & \longrightarrow & Df_*[\sigma] \end{array}$$

commute, so we can then define a map $f_! D\underline{\underline{I}}^\bullet \to Df_* \underline{\underline{I}}^\bullet$ (since $\underline{\underline{I}}^\bullet \in D^b_f(X)$, each $\underline{\underline{I}}^i$ is a finite direct sum of $[\sigma]$'s). The proof of the theorem is complete, then, once we have shown that $f_! D[\sigma] \to Df_*[\sigma]$ is a quasi-isomorphism.

Let $\sigma \in X$ and $\gamma \leq f(\sigma)$, $\dim \sigma = n$, $\dim \gamma = m$, and consider the double complex which has in position $(-i,j)$ the vector space $\bigoplus_{\substack{\mu^j \leq \lambda^i \leq \sigma \\ f(\mu^j) = \gamma}} \mathbb{Q}_{\mu^j, \lambda^i}$ (each $\mathbb{Q}_{\mu,\lambda}$ is just a copy of $\mathbb{Q}$), where

$\mathbb{Q}_{\mu^j,\lambda^i} \to \mathbb{Q}_{\mu^{j+1},\lambda^i}$ is multiplication by $[\mu^{j+1},\mu^j]$ and $\mathbb{Q}_{\mu^j,\lambda^i} \to \mathbb{Q}_{\mu^j,\lambda^{i-1}}$ is similarly multiplication by $[\lambda^i:\lambda^{i-1}]$. The $j^{\underline{th}}$ column is the direct sum of chain complexes $\bigoplus_{\substack{\mu^j \leq \sigma \\ f(\mu^j) = \gamma}} S^{\bullet *}_{\mu^j}$ where $S^{\bullet *}_{\mu^j}$ is the dual of the chain complex of lemma 3.2.3. Since $V \longmapsto V^*$ is an exact functor on the category of vector spaces, $S^{\bullet *}_{\mu^j}$ is exact for $j < n$. If $j = n$, i.e., $\mu^j = \sigma$, then $\gamma = f(\sigma)$. So for $\gamma < f(\sigma)$, all columns of the double complex are exact, hence it is quasi-isomorphic to 0. For $\gamma = f(\sigma)$, all columns are exact except the n^{th} one which consists only of $\mathbb{Q}_{\sigma,\sigma}$ in position $(-n,n)$. Then the double complex is quasi-isomorphic to the single complex with $\mathbb{Q}$ in degree 0 and 0 in other degrees, since an explicit quasi-isomorphism with the double complex consisting of $\mathbb{Q}$ in position $(-n,n)$ and 0 elsewhere, can be constructed.

Now look at the $-i^{\underline{\text{th}}}$ row. This is, in the notation of the beginning of the proof, $\bigoplus_{\lambda^i \leq \sigma} C^{\bullet}_{\gamma}(\lambda^i)$. But because of the exact sequence

$$0 \to f_!\,[\lambda^i](\gamma) \to C^0_{\gamma}(\lambda^i) \to C^1_{\gamma}(\lambda^i) \to \cdots$$

and corollary 1.2.3, we have that the double complex is quasi-isomorphic to

$$f_!\,[\sigma](\gamma) \to \bigoplus_{\lambda^{n-1} < \sigma} f_!\,[\lambda^{n-1}](\gamma) \to \bigoplus_{\lambda^{n-2} < \sigma} f_!\,[\lambda^{n-2}](\gamma) \to \cdots$$

the complex starting in degree $m - n$. Then for $\gamma \neq f(\sigma)$, $f_!\,D[\sigma](\gamma)$ is exact and for $\gamma = f(\sigma)$, $f_!\,D[\sigma](\gamma)$ is quasi-isomorphic to a chain complex with $\mathbb{Q}$ in degree $-m$ and 0 elsewhere. Clearly $Df_*[\sigma](f(\sigma))$ is quasi-isomorphic to $\mathbb{Q}$ in degree $-m$, 0 elsewhere, and for $\gamma < f(\sigma)$, $Df_*[\sigma](\gamma)$ is exact by lemma 3.2.3.

To see that $f_!D[\sigma] \to Df_*[\sigma]$ is a quasi-isomorphism, then, we just need to check it over $\gamma = f(\sigma)$. We have

$$\begin{tikzcd} f_![\sigma](\gamma) \arrow[r, "\partial^{-n}"] & \bigoplus_{\lambda^{n-1} < \sigma} f_![\lambda^{n-1}](\gamma) \arrow[r, "\partial^{-n+1}"] & \cdots \arrow[r, "\partial^{-m-1}"] & \bigoplus_{\lambda^m < \sigma} f_![\lambda^m](\gamma) \arrow[d, "\alpha"] \\ & & & {[\gamma](\gamma)} \end{tikzcd}$$

which we need to show is exact. α is surjective since we can always find a $\lambda^m < \sigma$ with $f(\lambda^m) = \gamma$, so the complex is exact at $[\gamma](\gamma)$. Then $\dim \ker(\alpha) = \dim(\bigoplus_{\lambda^m < \sigma} f_![\lambda^m](\gamma)) - \dim([\gamma](\gamma)) = \dim \mathrm{Im}(\partial^{-m-1})$ since $f_!D[\sigma](\gamma)$ and $Df_*[\sigma](\gamma)$ have the same cohomology, and this shows that the complex is exact at $\bigoplus_{\lambda^m < \sigma} f_![\lambda^m](\gamma)$. The proof is now complete.

<u>The functor $f^!$</u>

Given $f : X \to Y$ a cellular map, we define $f^! : D^b(Y) \to D^b(X)$ to be $f^!\underline{\underline{A}}^\bullet = Df^*D\underline{\underline{A}}^\bullet$ for $\underline{\underline{A}}^\bullet \in D^b_f(X)$. Although this definition is cumbersome to work with, there are simpler descriptions of $f^!$ in a number of special cases.

Theorem 3.3.3: For $f : X \to Y$ a cellular map, $f^! \mathbb{D}^\bullet_Y \overset{qi}{=} \mathbb{D}^\bullet_X$.

Proof: $f^! \mathbb{D}^\bullet_Y = Df^* D\mathbb{D}^\bullet_Y \overset{qi}{=} Df^* \underline{\underline{Q}}_Y = D\underline{\underline{Q}}_X = \mathbb{D}^\bullet_X$.

Theorem 3.3.4: If γ is an m-cell in Y and $f : X \to Y$ is a cellular map, then $f^![\gamma]$ is quasi-isomorphic to the complex

$$\cdots \longrightarrow \overset{\text{deg.m-k}}{\bigoplus_{\tau^k \in f^{-1}(\gamma)}} [\tau^k] \longrightarrow \overset{\text{deg.m-k+1}}{\bigoplus_{\tau^{k-1} \in f^{-1}(\gamma)}} [\tau^{k-1}] \longrightarrow \cdots$$

where maps between components are given by multiplication by $[\tau^{k+1}:\tau^k]$ as usual.

Proof: It follows from lemma 3.2.3 that $D[\gamma]$ is quasi-isomorphic to $(\gamma)[m]$ where (γ) is the sheaf with Q on σ and 0 elsewhere. Then $f^![\gamma] = Df^*D[\gamma] \overset{qi}{=} Df^*((\gamma)[m]) \overset{qi}{=} D(\underline{\underline{Q}}_{f^{-1}(\gamma)}[m])$ where $\underline{\underline{Q}}_{f^{-1}(\gamma)}$ is the constant sheaf on $f^{-1}(\gamma)$ and 0 elsewhere. This is the complex which has in degree $m-k$, $\bigoplus_{\tau^k \in X} \underline{\underline{\mathrm{Hom}}}(\underline{\underline{Q}}_{f^{-1}(\gamma)}, [\tau^k])$

$= \bigoplus_{\tau^k \in f^{-1}(\gamma)} \underline{\underline{\mathrm{Hom}}}(\underline{\underline{Q}}_{f^{-1}(\gamma)}, [\tau^k]) = \bigoplus_{\tau^k \in f^{-1}(\gamma)} [\tau^k]$. The component maps are easily seen to be given by multiplication by $[\tau^{k+1}:\tau^k]$.

Theorem 3.3.5: If $i : X \to Y$ is an inclusion, $\underline{\underline{I}}^\bullet \in D^b_f(Y)$ an injective complex with $\underline{\underline{I}}^i = \bigoplus_{\gamma \in Y} [\gamma]^{V^i_\gamma}$, then $f^! \underline{\underline{I}}^\bullet$ is the complex of sheaves with $\bigoplus_{\gamma \in X} [\gamma]^{V^i_\gamma}$ in degree i and with component maps

$[\gamma]^{V^i_\gamma} \to [\mu]^{V^i_\mu}$ being given by the same map as before.

<u>Proof</u>: Since different choices of orientations on the cells of X and Y result in the same dualizing complexes up to (chain) isomorphism (proposition 3.2.1), it follows that $i^!\underline{\underline{I}}^\bullet$ is independent of the orientations chosen, as well. We can then assume that i preserves orientation on each cell of X.

For $\tau \in Y$ we will let $[\tau]_Y$ be the elementary injective on Y and $[\tau]_X$ the elementary injective on X if $\tau \in X$, or 0 if $\tau \notin X$. It follows from lemma 1.6.1 that if $\gamma \leq \tau \leq \sigma$ are in Y and $\gamma,\sigma \in X$, then $\tau \in X$ (apply the lemma to the map $\overline{\sigma} \cap X \to \overline{\sigma}$, closure taken in Y, and a point $y \in |\gamma|$). The correspondence $[\tau]_Y \longmapsto [\tau]_X$ then defines a functor from injective sheaves with finite dimensional stalks on Y to injective sheaves on X which, given cells $\tau \leq \sigma$ in X, sends $r_Y : [\sigma]_Y \to [\tau]_Y$ to the map $r_X : [\sigma]_X \to [\tau]_X$ which has $r_X(\tau) = r_Y(\tau)$. This induces a functor $\underline{\underline{I}}^\bullet \longmapsto \underline{\underline{I}}^\bullet_X$ on injective complexes, and the theorem claims that $i^!\underline{\underline{I}}^\bullet \overset{qi}{=\!=} \underline{\underline{I}}^\bullet_X$. We will construct a quasi-isomorphism $i^*D\underline{\underline{I}}^\bullet \to D\underline{\underline{I}}^\bullet_X$; the theorem then follows by dualizing each side and applying theorem 3.2.4.

We first define $i^*D[\tau]_Y \to D[\tau]_X$. If $\tau \notin X$ then $D[\tau]_X = 0$, so this is the 0-map. If $\tau \in X$, $\dim \tau = n$, we want to define a chain map

$$\begin{array}{ccccc} & \text{deg.}-n & & \text{deg.}-n+1 & & \text{deg.}-n+2 \\ i^*D[\tau]_Y = & i^*[\tau]_Y & \longrightarrow \bigoplus_{\gamma^{n-1}<\tau} & i^*[\gamma^{n-1}]_Y & \longrightarrow \bigoplus_{\gamma^{n-2}<\tau} & i^*[\gamma^{n-2}]_Y \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow \\ D[\tau]_X = & [\tau]_X & \longrightarrow \bigoplus_{\gamma^{n-1}<\tau} & [\gamma^{n-1}]_X & \longrightarrow \bigoplus_{\gamma^{n-2}<\tau} & [\gamma^{n-2}]_X \longrightarrow \cdots \end{array} .$$

But for $\gamma \in X$, $i^*[\gamma]_Y = [\gamma]_X$ since the closure of γ in X is the intersection of X with the closure of γ in Y , so we can take $i^*[\gamma]_Y \to [\gamma]_X$ to be the identity. It easily follows that this is a chain map from the fact that for $\gamma^{i-1} < \gamma^i \le \tau \in X$, if $\gamma^{i-1} \in X$, then $\gamma^i \in X$.

To see that $i^*D[\tau]_Y \to D[\tau]_X$ is a quasi-isomorphism, recall that $D[\tau]_Y \overset{qi}{=} (\tau)[n]$, where (τ) is the sheaf which is $\mathbb{Q}$ over τ and 0 elsewhere (this follows from lemma 3.2.3). Then for $\tau \notin X$, $D[\tau]_X = 0$ and $i^*D[\tau]_Y \overset{qi}{=} i^*(\tau)[n] = 0$; for $\tau \in X$, $D[\tau]_X \overset{qi}{=} (\tau)[n]$ and $i^*D[\tau]_Y \overset{qi}{=} i^*(\tau)[n] = (\tau)[n]$. The map is clearly a quasi-isomorphism over the cell τ , so this shows it is a quasi-isomorphism.

We now want to check that for a sheaf map $[\sigma]_Y \to [\tau]_Y$ with $\tau \le \sigma$, the following commutes:

$$\begin{array}{ccc} i^* D[\tau]_Y & \longrightarrow & D[\tau]_X \\ \downarrow & & \downarrow \\ i^* D[\sigma]_Y & \longrightarrow & D[\sigma]_X \end{array} .$$

If σ and τ are both in X, this can be easily verified. If $\sigma \notin X$, then $D[\sigma]_X = 0$, and if $\sigma \in X$ but $\tau \notin X$, then no face of τ can be in X, so $i^* D[\tau]_Y = 0$. Hence the diagram always commutes. This shows that we have a map $i^* D\underline{\underline{I}}^\bullet \to D\underline{\underline{I}}_X^\bullet$ between double complexes where each slice $i^* D\underline{\underline{I}}^j \to D\underline{\underline{I}}_X^j$ is a quasi-isomorphism, so the map is a quasi-isomorphism. This completes the proof.

§3.4 Identities

Theorem 3.4.1: For $i : X \hookrightarrow Y$ an inclusion with X open in Y, $f^! \underline{\underline{A}}^\bullet \overset{qi}{=} f^* \underline{\underline{A}}^\bullet$ for $\underline{\underline{A}}^\bullet \in D_f^b(Y)$.

Proof: Take $\underline{\underline{A}}^\bullet$ to be injective. Then the result follows from the definition of f^* and theorem 3.3.5.

Note that for $f : X \to Y$ a cellular map, we have a natural map $f_! \underline{\underline{A}} \to f_* \underline{\underline{A}}$ for $\underline{\underline{A}}$ a single sheaf which over $\tau \in Y$ takes a section $\alpha \in \Gamma(f^{-1}(\tau), \underline{\underline{A}})$ to the section of $\underline{\underline{A}}$ on $f^{-1}(st(\tau))$ which restricts to $p_{\tau,\sigma}^{f_! \underline{\underline{A}}}(\alpha)$ on $f^{-1}(\sigma)$ for $\sigma \geq \tau$. This extends to a map $R^\bullet f_! \underline{\underline{A}}^\bullet \to R^\bullet f_* \underline{\underline{A}}^\bullet$ in $D^b(Y)$.

Theorem 3.4.2: For $f : X \to Y$ a proper map the natural map $R^{\bullet}f_{!}\underline{\underline{A}}^{\bullet} \to R^{\bullet}f_{*}\underline{\underline{A}}^{\bullet}$ is a quasi-isomorphism for $\underline{\underline{A}}^{\bullet} \in D^{b}(X)$.

Proof: If f is proper, and if $\sigma \in X$ and $\tau \le f(\sigma)$, then there is a cell $\gamma \le \sigma$ where $f(\gamma) = \tau$. To see this, restrict f to the map $f' : \overline{\sigma} \to \overline{f(\sigma)}$, and take a compact neighborhood C of a point $y \in |\tau|$. f' will still be proper, so $f'^{-1}(C)$ is compact. Then $f'f'^{-1}(C)$ is compact and contains $C \cap |f(\sigma)|$, so $f'f'^{-1}(C)$ contains $y \in \overline{C \cap |f(\sigma)|}$. Then $f'^{-1}(y)$ is non-empty, and the statement is shown.

We claim that for a single sheaf $\underline{\underline{A}}$, the map $f_{!}\underline{\underline{A}} \to f_{*}\underline{\underline{A}}$ is an isomorphism. For $\tau \in Y$, the map $f_{!}\underline{\underline{A}}(\tau) \to f_{*}\underline{\underline{A}}(\tau)$ is clearly one-to-one (if it takes α to $\tilde{\alpha}$, then $\alpha \in \Gamma(f^{-1}(\tau),\underline{\underline{A}})$ is just the restriction of $\tilde{\alpha} \in \Gamma(f^{-1}(st(\tau)),\underline{\underline{A}})$ to $f^{-1}(\tau))$. To see that the map is surjective, let $\tilde{\alpha} \in \Gamma(f^{-1}(st(\tau)),\underline{\underline{A}})$. Since $f^{-1}(y)$ is compact for $y \in |Y|$, for $\gamma \ge \tau$, $\tilde{\alpha}$ restricted to $f^{-1}(\gamma)$ is an element of $f_{!}\underline{\underline{A}}(\gamma)$. Given $\mu > \gamma \ge \tau$, it follows from the fact that every cell of $f^{-1}(\mu)$ has a face in $f^{-1}(\gamma)$ (by the statement at the beginning of the proof), that $p_{\gamma,\mu}^{f_{!}\underline{\underline{A}}}$ maps $\tilde{\alpha}|_{f^{-1}(\gamma)}$ to $\tilde{\alpha}|_{f^{-1}(\mu)}$. Then $f_{!}\underline{\underline{A}}(\tau) \to f_{*}\underline{\underline{A}}(\tau)$ takes $\tilde{\alpha}|_{f^{-1}(\tau)}$ to $\tilde{\alpha}$, so the map is surjective. Therefore the map is an isomorphism, and the theorem follows immediately from this.

Theorem 3.4.3: Let $g : X \to Y$, $f : Y \to Z$ and $fg : X \to Z$ be cellular maps. Then,

(i) $R^\bullet(fg)_*\underline{\underline{A}}^\bullet \overset{qi}{\equiv} R^\bullet f_* R^\bullet g_* \underline{\underline{A}}^\bullet$ for $\underline{\underline{A}}^\bullet \in D^b(X)$.

(ii) $R^\bullet(fg)_!\underline{\underline{A}}^\bullet \overset{qi}{\equiv} R^\bullet f_! R^\bullet g_! \underline{\underline{A}}^\bullet$ for $\underline{\underline{A}}^\bullet \in D^b_f(X)$ and f, g, and fg fibred cellular maps.

(iii) $(fg)^*\underline{\underline{A}}^\bullet \overset{qi}{\equiv} g^* f^* \underline{\underline{A}}^\bullet$ for $\underline{\underline{A}}^\bullet \in D^b(Z)$.

(iv) $(fg)^!\underline{\underline{A}}^\bullet \overset{qi}{\equiv} g^! f^! \underline{\underline{A}}^\bullet$ for $\underline{\underline{A}}^\bullet \in D^b_f(Z)$.

Proof: (i) It's easily verified that $(fg)_*\underline{\underline{A}} = f_* g_* \underline{\underline{A}}$ for $\underline{\underline{A}}$ a sheaf. Then (i) follows from the fact that g_* of an injective sheaf is injective.

(ii) $R^\bullet(fg)_!\underline{\underline{A}}^\bullet \overset{qi}{\equiv} DR^\bullet(fg)_* D\underline{\underline{A}}^\bullet \overset{qi}{\equiv} DR^\bullet f_* R^\bullet g_* D\underline{\underline{A}}^\bullet$ $\overset{qi}{\equiv} DR^\bullet f_* DDR^\bullet g_* D\underline{\underline{A}}^\bullet \overset{qi}{\equiv} R^\bullet f_! R^\bullet g_! \underline{\underline{A}}^\bullet$.

(iii) trivial

(iv) $(fg)^!\underline{\underline{A}}^\bullet \overset{qi}{\equiv} D(fg)^* D\underline{\underline{A}}^\bullet \overset{qi}{\equiv} Dg^* f^* D\underline{\underline{A}}^\bullet \overset{qi}{\equiv} Dg^* DDf^* D\underline{\underline{A}}^\bullet$ $\overset{qi}{\equiv} g^! f^! \underline{\underline{A}}^\bullet$.

Theorem 3.4.4: For $\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet, \underline{\underline{C}}^\bullet \in D^b(X)$, $R^\bullet \underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^\bullet, R^\bullet\underline{\underline{\mathrm{Hom}}}(\underline{\underline{B}}^\bullet, \underline{\underline{C}}^\bullet)) \overset{qi}{\equiv} R^\bullet\underline{\underline{\mathrm{Hom}}}(\underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet, \underline{\underline{C}}^\bullet)$.

Proof: We begin with the case where $\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet$, and $\underline{\underline{C}}^\bullet$ are single elementary injectives $[\sigma]^U$, $[\tau]^V$, and $[\lambda]^W$, respectively. $\underline{\underline{\mathrm{Hom}}}([\sigma]^U, \mathrm{Hom}([\tau]^V, [\lambda]^W))$ is 0 unless $\lambda \le \sigma$ and $\lambda \le \tau$, in which case the sheaf is $[\lambda]^{\mathrm{Hom}(U,\mathrm{Hom}(V,W))}$. Similarly, $\mathrm{Hom}([\sigma]^U \otimes [\tau]^V, [\lambda]^W)$ is 0 unless $\lambda \le \sigma$ and $\lambda \le \tau$, in which case the sheaf is $[\lambda]^{\mathrm{Hom}(U\otimes V,W)}$. For any vector spaces U, V, and W , there is a natural isomorphism $\mathrm{Hom}(U,\mathrm{Hom}(V,W)) \to \mathrm{Hom}(U\otimes V,W)$, and this gives us an isomorphism $\underline{\underline{\mathrm{Hom}}}([\sigma]^U, \underline{\underline{\mathrm{Hom}}}([\tau]^V, [\lambda]^W)) \to \underline{\underline{\mathrm{Hom}}}([\sigma]^U \otimes [\tau]^V, [\lambda]^W)$.

If $\underline{\underline{I}}$, $\underline{\underline{J}}$, and $\underline{\underline{K}}$ are injective sheaves, we then have a natural isomorphism $\underline{\underline{Hom}}(\underline{\underline{I}},\underline{\underline{Hom}}(\underline{\underline{J}},\underline{\underline{K}})) \to \underline{\underline{Hom}}(\underline{\underline{I}}\otimes\underline{\underline{J}},\underline{\underline{K}})$ acquired by breaking each injective up into a direct sum of elementary injectives.

Now let $\underline{\underline{I}}^\bullet$, $\underline{\underline{J}}^\bullet$, and $\underline{\underline{K}}^\bullet$ be injective complexes. Then $\underline{\underline{Hom}}^\bullet(\underline{\underline{I}}^\bullet,\underline{\underline{Hom}}^\bullet(\underline{\underline{J}}^\bullet,\underline{\underline{K}}^\bullet))$ and $\underline{\underline{Hom}}^\bullet(\underline{\underline{I}}^\bullet \otimes \underline{\underline{J}}^\bullet,\underline{\underline{K}}^\bullet)$ are represented by triple complexes, and in each position (i,j,k) we have an isomorphism of sheaves from the first triple complex to the second. It can easily be verified that these isomorphisms commute with boundary maps. For example, to check that the diagram

$$\begin{array}{ccc} \underline{\underline{Hom}}(\underline{\underline{I}}^i,\underline{\underline{Hom}}(\underline{\underline{J}}^j,\underline{\underline{K}}^k)) & \longrightarrow & \underline{\underline{Hom}}(\underline{\underline{I}}^i \otimes \underline{\underline{J}}^j,\underline{\underline{K}}^k) \\ \downarrow & & \downarrow \\ \underline{\underline{Hom}}(\underline{\underline{I}}^i,\underline{\underline{Hom}}(\underline{\underline{J}}^j,\underline{\underline{K}}^{k+1})) & \longrightarrow & \underline{\underline{Hom}}(\underline{\underline{I}}^i \otimes \underline{\underline{J}}^j,\underline{\underline{K}}^{k+1}) \end{array}$$

commutes, break $\underline{\underline{I}}^i$, $\underline{\underline{J}}^j$, $\underline{\underline{K}}^k$, and $\underline{\underline{K}}^{k+1}$ up into elementary injectives, and note that the analogous diagram of vector spaces commutes.

We then have an isomorphism of triple complexes, so the induced map on single complexes is an isomorphism, and the theorem follows.

<u>Theorem 3.4.5</u>: For $\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet \in D^b_f(X)$, $R^\bullet\underline{\underline{Hom}}(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet) \overset{qi}{=} D(\underline{\underline{A}}^\bullet \otimes D\underline{\underline{B}}^\bullet)$.

<u>Proof</u>: This follows from theorems 3.4.4 and 3.2.4 by replacing $\underline{\underline{B}}^\bullet$ with $D\underline{\underline{B}}^\bullet$ and $\underline{\underline{C}}^\bullet$ with $\underline{\underline{\mathbb{D}}}^\bullet_X$.

Theorem 3.4.6: For $\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet \in D_f^b(X)$,
$R^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet) \overset{qi}{=} R^\bullet \underline{\underline{Hom}}(D\underline{\underline{B}}^\bullet, D\underline{\underline{A}}^\bullet)$.

Proof: $R^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet) \overset{qi}{=} D(\underline{\underline{A}}^\bullet \otimes D\underline{\underline{B}}^\bullet)$ (theorem 3.4.5)
$\overset{qi}{=} D(D\underline{\underline{B}}^\bullet \otimes \underline{\underline{A}}^\bullet) \overset{qi}{=} D(D\underline{\underline{B}}^\bullet \otimes DD\underline{\underline{A}}^\bullet) \overset{qi}{=} R^\bullet \underline{\underline{Hom}}(D\underline{\underline{B}}^\bullet, D\underline{\underline{A}}^\bullet)$.

Theorem 3.4.7: Given $f : X \to Y$ a cellular map and $\underline{\underline{A}}^\bullet \in D^b(Y)$, $\underline{\underline{B}}^\bullet \in D^b(X)$, then $R^\bullet f_* R^\bullet \underline{\underline{Hom}}(f^* \underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet) \overset{qi}{=} R^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, R^\bullet f_* \underline{\underline{B}}^\bullet)$.

Proof: Let $\underline{\underline{A}}^\bullet = [\sigma]^V$ and $\underline{\underline{B}}^\bullet = [\tau]^W$. $f^*[\sigma]^V$ is the constant sheaf V on $f^{-1}(\overline{\sigma})$ and 0 elsewhere, so $\mathrm{Hom}(f^*[\sigma]^V, [\tau]^W) = 0$ if $\tau \nleq f^{-1}(\overline{\sigma})$, i.e., if $f(\tau) \nleq \sigma$. If $f(\tau) \leq \sigma$ then it's easily checked that $\underline{\underline{Hom}}(f^*[\sigma]^V, [\tau]^W) = [\tau]^{\mathrm{Hom}(V,W)}$. Then $f_* \underline{\underline{Hom}}(f^*[\sigma]^V, [\tau]^W)$ is 0 for $f(\tau) \nleq \sigma$ and $[f(\tau)]^{\mathrm{Hom}(V,W)}$ for $f(\tau) \leq \sigma$. The same is true for $\underline{\underline{Hom}}([\sigma]^V, f_*[\tau]^W)$, so we have an isomorphism $f_* \underline{\underline{Hom}}(f^*[\sigma]^V, [\tau]^W) \to \underline{\underline{Hom}}([\sigma]^V, f_*[\tau]^W)$ which gives rise to an isomorphism $f_* \underline{\underline{Hom}}(f^* \underline{\underline{I}}, \underline{\underline{J}}) \to \underline{\underline{Hom}}(\underline{\underline{I}}, f_* \underline{\underline{J}})$ for injective sheaves $\underline{\underline{I}}$ and $\underline{\underline{J}}$. It can be easily verified that these isomorphisms commute with maps induced by boundary maps in the two injective complexes $\underline{\underline{I}}^\bullet$ and $\underline{\underline{J}}^\bullet$, so we have an isomorphism $f_* \underline{\underline{Hom}}^\bullet(f^* \underline{\underline{I}}^\bullet, \underline{\underline{J}}^\bullet) \to \underline{\underline{Hom}}^\bullet(\underline{\underline{I}}^\bullet, f_* \underline{\underline{J}}^\bullet)$, or $R^\bullet f_* R^\bullet \underline{\underline{Hom}}(f^* \underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet) \to R^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, R^\bullet f_* \underline{\underline{B}}^\bullet)$ where there are injective resolutions $\underline{\underline{A}}^\bullet \to \underline{\underline{I}}^\bullet$ and $\underline{\underline{B}}^\bullet \to \underline{\underline{J}}^\bullet$.

Theorem 3.4.8 (Verdier Duality): For $f : X \to Y$ a fibred cellular map, $\underline{\underline{A}}^\bullet \in D_f^b(X)$, and $\underline{\underline{B}}^\bullet \in D_f^b(Y)$, we have
$R^\bullet f_* R^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, f^! \underline{\underline{B}}^\bullet) \overset{qi}{=} R^\bullet \underline{\underline{Hom}}(Rf_! \underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet)$.

Proof: $R^\bullet f_* R^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, f^! \underline{\underline{B}}^\bullet) \overset{qi}{=} R^\bullet f_* R^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, Df^* D\underline{\underline{B}}^\bullet)$

$\overset{qi}{=} R^\bullet f_* R^\bullet \underline{\underline{Hom}}(f^* D\underline{\underline{B}}^\bullet, D\underline{\underline{A}}^\bullet)$ (by theorem 3.4.6 and biduality)

$\overset{qi}{=} R^\bullet \underline{\underline{Hom}}(D\underline{\underline{B}}^\bullet, R^\bullet f_* D\underline{\underline{A}}^\bullet)$ (theorem 3.4.7)

$\overset{qi}{=} R^\bullet \underline{\underline{Hom}}(DR^\bullet f_* D\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet) \overset{qi}{=} R^\bullet \underline{\underline{Hom}}(R^\bullet f_! \underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet)$ (theorem 3.3.2).

Theorem 3.4.9: For $f : X \to Y$ a fibred cellular map, $\underline{\underline{A}}^\bullet \in D^b_f(Y)$, and $\underline{\underline{B}}^\bullet \in D^b_f(X)$, then $\underline{\underline{A}}^\bullet \otimes R^\bullet f_! \underline{\underline{B}}^\bullet \overset{qi}{=} R^\bullet f_!(f^* \underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet)$.

Proof: $\underline{\underline{A}}^\bullet \otimes R^\bullet f_! \underline{\underline{B}}^\bullet \overset{qi}{=} DR^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, DR^\bullet f_! \underline{\underline{B}}^\bullet)$ (theorem 3.4.5)

$\overset{qi}{=} DR^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, R^\bullet f_* D\underline{\underline{B}}^\bullet) \overset{qi}{=} DR^\bullet f_* R^\bullet \underline{\underline{Hom}}(f^* \underline{\underline{A}}^\bullet, D\underline{\underline{B}}^\bullet)$ (theorem 3.4.7)

$\overset{qi}{=} R^\bullet f_! DR^\bullet \underline{\underline{Hom}}(f^* \underline{\underline{A}}^\bullet, D\underline{\underline{B}}^\bullet) \overset{qi}{=} R^\bullet f_!(f^* \underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet)$.

Theorem 3.4.10: Given $f : X \to Y$ a cellular map and $\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet \in D^b(Y)$, we have $f^*(\underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet) \overset{qi}{=} f^* \underline{\underline{A}}^\bullet \otimes f^* \underline{\underline{B}}^\bullet$.

Proof: Over $\sigma \in X$, $f^*(\underline{\underline{A}}^i \otimes \underline{\underline{B}}^j)$ and $f^* \underline{\underline{A}}^i \otimes f^* \underline{\underline{B}}^j$ are both $\underline{\underline{A}}^i(f(\sigma)) \otimes \underline{\underline{B}}^j(f(\sigma))$, and the corestriction maps agree. The boundary maps of $\underline{\underline{A}}^\bullet$ and $\underline{\underline{B}}^\bullet$ clearly are compatible with the canonical isomorphisms $f^*(\underline{\underline{A}}^i \otimes \underline{\underline{B}}^j) \to f^* \underline{\underline{A}}^i \otimes f^* \underline{\underline{B}}^j$, so there is an isomorphism of double complexes inducing an isomorphism $f^*(\underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet) \to f^* \underline{\underline{A}}^\bullet \otimes f^* \underline{\underline{B}}^\bullet$.

Theorem 3.4.11: For $f : X \to Y$ a cellular map and $\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet \in D^b_f(Y)$, $f^! R^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet) \overset{qi}{=} R^\bullet \underline{\underline{Hom}}(f^* \underline{\underline{A}}^\bullet, f^! \underline{\underline{B}}^\bullet)$.

Proof: $R^\bullet \underline{\underline{Hom}}(f^* \underline{\underline{A}}^\bullet, f^! \underline{\underline{B}}^\bullet) \overset{qi}{=} R^\bullet \underline{\underline{Hom}}(f^* \underline{\underline{A}}^\bullet, Df^* D\underline{\underline{B}}^\bullet)$

$\overset{qi}{=} D(f^* \underline{\underline{A}}^\bullet \otimes f^* D\underline{\underline{B}}^\bullet)$ (by theorem 3.4.5)

$\overset{qi}{=} Df^*(\underline{\underline{A}}^\bullet \otimes D\underline{\underline{B}}^\bullet) \overset{qi}{=} Df^* DR^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet) \overset{qi}{=} f^! R^\bullet \underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet)$.

One nice property of cell complexes is that there is a natural way of forming a product cell complex $X \times Y$ from complexes X and Y.

<u>Theorem 3.4.12</u>: Let $X = \{\sigma_i\}$ and $Y = \{\tau_j\}$ be cell complexes. Then

(i) the sets $\{\sigma_i \times \tau_j\}$ give $|X| \times |Y|$ the structure of a cell complex;

(ii) The projection maps $\pi_1 : |X| \times |Y| \to |X|$ and $\pi_2 : |X| \times |Y| \to |Y|$ are fibred cellular maps, and for 0-cells $v \in X$, $w \in Y$, the inclusions $i_v : |Y| \hookrightarrow \{v\} \times |Y| \hookrightarrow |X| \times |Y|$ and $j_w : |X| \hookrightarrow |X| \times \{w\} \hookrightarrow |X| \times |Y|$ are fibred cellular maps.

<u>Proof</u>: $\{\sigma_i \times \tau_j\}$ clearly decomposes $|X| \times |Y|$ into open topological cells. Let $|\tilde{X}|$ and $|\tilde{Y}|$ be the one-point compactifications of $|X|$ and $|Y|$, respectively. Then the one-point compactification of $|X| \times |Y|$ can be formed by
$\widetilde{|X| \times |Y|} = |\tilde{X}| \times |\tilde{Y}| / (\{\infty\} \times |\tilde{Y}| \cup |\tilde{X}| \times \{\infty\})$. We want to show that this is a regular CW complex. To do this, we must show that $\partial(|\sigma| \times |\tau|)$ (boundary taken in $\widetilde{|X| \times |Y|}$) lies in the $(i+j-1)$-skeleton where $\dim \sigma = i$, $\dim \tau = j$, and that the pair $(\overline{|\sigma| \times |\tau|}, \partial(|\sigma| \times |\tau|))$ (closure and boundary taken in $\widetilde{|X| \times |Y|}$) is homeomorphic to (B^{i+j}, S^{i+j-1}).

We will assume that the closures of $|\sigma|$ and $|\tau|$ in $|X|$ and $|Y|$, respectively, are not compact, as the other cases are similar, but easier. Note that in $|\tilde{X}| \times |\tilde{Y}|$, the closure of $|\sigma| \times |\tau|$ is

$|\bar{\sigma}| \times |\bar{\tau}|$, and the boundary of $|\sigma| \times |\tau|$ is $\partial(|\sigma| \times |\tau|) =$
$(\partial|\sigma|) \times \ |\bar{\tau}| \cup |\bar{\sigma}| \times (\partial|\tau|)$
$= \bigcup\{|\sigma'| \times |\tau'| \mid \sigma' \le \sigma, \tau' \le \tau, \dim\sigma' + \dim\tau' < i+j\}$ (all closures and boundaries of $|\sigma|$ and $|\tau|$ taken in $|\tilde{X}|$ and $|\tilde{Y}|$). $\overline{|\sigma| \times |\tau|}$ will always refer to closure in $\widetilde{|X| \times |Y|}$, and $\tilde{\partial}(|\sigma| \times |\tau|)$ will refer to the boundary in $\widetilde{|X| \times |Y|}$. If $\pi : |\tilde{X}| \times |\tilde{Y}| \to \widetilde{|X| \times |Y|}$ is the identification map, then $\overline{|\sigma| \times |\tau|} = \pi(|\bar{\sigma}| \times |\bar{\tau}|)$ and $\tilde{\partial}(|\sigma| \times |\tau|) = \pi \circ \partial(|\sigma| \times |\tau|)$, so it's clear from the above description of $\partial(|\sigma| \times |\tau|)$ that $\tilde{\partial}(|\sigma| \times |\tau|)$ lies in the (i+j-1)-skeleton. To see the second statement, note that the pair $(\overline{|\sigma| \times |\tau|}, \tilde{\partial}(|\sigma| \times |\tau|))$ is the pair $(|\bar{\sigma}| \times |\bar{\tau}|, \partial(|\sigma| \times |\tau|))$ with the two closed cells $|\bar{\sigma}| \times \{\infty\}$ and $\{\infty\} \times |\bar{\tau}|$ of $\partial(|\sigma| \times |\tau|)$ identified to a single point. Since the union of the two cells is connected (they both contain (∞,∞)), we can identify one to a point, and then identify the other to a point. In general, if γ is a cell and $\bar{\gamma}$ is a compact cell complex, and if a closed cell on $\partial\gamma$ is identified to a point, then $(|\bar{\gamma}|, \partial|\gamma|)$ is still a pair of spaces homeomorphic to (B^n, S^{n-1}) . Then the result of identifying $|\bar{\sigma}| \times \{\infty\}$ to a point is a pair of spaces homeomorphic to (B^{i+j}, S^{i+j-1}) . Under this identification, a closed cell $|\overline{\sigma'}| \times |\overline{\tau'}|$ is unchanged if $\infty \notin |\overline{\tau'}|$ and is $|\overline{\sigma'}| \times |\overline{\tau'}| / |\overline{\sigma'}| \times \{\infty\}$ if $\infty \in |\overline{\tau'}|$, which is homeomorphic to (B^n, S^{n-1}) . Then $(|\bar{\sigma}| \times |\bar{\tau}|, \partial(|\sigma| \times |\tau|))$ is a regular cell complex after identifying $|\bar{\sigma}| \times \{\infty\}$ to a point, and hence is still homeomorphic to (B^{i+j}, S^{i+j-1}) after identifying $\{\infty\} \times |\bar{\tau}|$ to a point.

(ii) These maps are clearly cellular maps. Since the cohomology with compact support of a compact cell possibly minus a point on its boundary is 0 except in dimension 0, the maps π_1 and π_2 are fibred. i_v and j_w are clearly fibred.

If $\underline{\underline{A}}^{\bullet} \in D^b(X)$ and $\underline{\underline{B}}^{\bullet} \in D^b(Y)$, we define $\underline{\underline{A}}^{\bullet} \times \underline{\underline{B}}^{\bullet} \in D^b(X \times Y)$ to be the single complex associated to the double complex with $\underline{\underline{A}}^i \times \underline{\underline{B}}^j$ in degree (i,j), where $\underline{\underline{A}}^i \times \underline{\underline{B}}^j$ is the sheaf with $\underline{\underline{A}}^i(\sigma) \otimes \underline{\underline{B}}^j(\tau)$ over $\sigma \times \tau$, and $p^{\underline{\underline{A}}^i \times \underline{\underline{B}}^j}_{\sigma' \times \tau', \sigma\times\tau} = p^{\underline{\underline{A}}^i}_{\sigma',\sigma} \otimes p^{\underline{\underline{B}}^j}_{\tau',\tau}$. It's clear that $\underline{\underline{A}}^{\bullet} \times \underline{\underline{B}}^{\bullet} = (\pi_1^* \underline{\underline{A}}^{\bullet}) \otimes (\pi_2^* \underline{\underline{B}}^{\bullet})$, so in particular, $\times$ is functorial on $D^b(X)$ and $D^b(Y)$. If $\underline{\underline{I}}^{\bullet} \in D^b(X)$ and $\underline{\underline{J}}^{\bullet} \in D^b(Y)$ are injective complexes, then $\underline{\underline{I}}^{\bullet} \times \underline{\underline{J}}^{\bullet}$ is an injective complex since $[\sigma]^V \times [\tau]^W = [\sigma \times \tau]^{V\otimes W}$.

Theorem 3.4.13: If $\pi_2 : X \times Y \to Y$ is the projection onto the second factor, and $\underline{\underline{A}}^{\bullet}, \underline{\underline{B}}^{\bullet} \in D^b(Y)$, then $\pi_2^* R^{\bullet}\underline{\underline{Hom}}(\underline{\underline{A}}^{\bullet}, \underline{\underline{B}}^{\bullet})$ $\underline{\underline{qi}}$ $R^{\bullet}\underline{\underline{Hom}}(\pi_2^*\underline{\underline{A}}^{\bullet}, \pi_2^*\underline{\underline{B}}^{\bullet})$.

Proof: We will assume that $\underline{\underline{A}}^{\bullet} = \underline{\underline{J}}^{\bullet}$ and $\underline{\underline{B}}^{\bullet} = \underline{\underline{K}}^{\bullet}$ are injective complexes.

Let $\underline{\underline{Q}}_X \to \underline{\underline{I}}^{\bullet}$ be an injective resolution of the constant sheaf on X. Then we have an injection resolution $\pi_2^* K^{\bullet} = \underline{\underline{Q}}_X \times \underline{\underline{K}}^{\bullet} \to \underline{\underline{I}}^{\bullet} \times \underline{\underline{K}}^{\bullet}$, and an induced map from a double complex to a triple complex $\underline{\underline{Hom}}^{\bullet}(\pi_2^*\underline{\underline{J}}^{\bullet}, \pi_2^*\underline{\underline{K}}^{\bullet}) \to \underline{\underline{Hom}}^{\bullet}(\pi_2^*\underline{\underline{J}}^{\bullet}, \underline{\underline{I}}^{\bullet} \times \underline{\underline{K}}^{\bullet}) = R^{\bullet}\underline{\underline{Hom}}(\pi_2^*\underline{\underline{J}}^{\bullet}, \pi_2^*\underline{\underline{K}}^{\bullet})$.

We claim that this map is a quasi-isomorphism, and hence there is no need to resolve $\pi_2^*\underline{\underline{K}}^\bullet$ to compute $R^\bullet\underline{\underline{\mathrm{Hom}}}(\pi_2^*\underline{\underline{J}}^\bullet,\pi_2^*\underline{\underline{K}}^\bullet)$. To show this, it is necessary only to check it for $\underline{\underline{J}}^\bullet = [\sigma]^V$ and $\underline{\underline{K}}^\bullet = [\tau]^W$. If $\underline{\underline{I}}^i = \bigoplus_j [\gamma_j]^{U_{ij}}$, then we have

$$\underline{\underline{\mathrm{Hom}}}(\pi_2^*[\sigma]^V,\pi_2^*[\tau]^W) \to \underline{\underline{\mathrm{Hom}}}(\pi_2^*[\sigma]^V,\bigoplus_j[\gamma_j\times\tau]^{U_{0j}\otimes W})$$

$$\to \underline{\underline{\mathrm{Hom}}}(\pi_2^*[\sigma]^V,\bigoplus_j[\gamma_j\times\tau]^{U_{1j}\otimes W}) \to \cdots .$$

$\pi_2^*[\sigma]^V$ is V on cells $\mu\times\lambda$ with $\lambda\le\sigma$ and 0 elsewhere, so this chain complex is 0 if $\tau \nleq \sigma$, and

$$\pi_2^*[\tau]^{\mathrm{Hom}(V,W)} \to \bigoplus_j[\gamma_j\times\tau]^{\mathrm{Hom}(V,U_{0j}\otimes W)} \to \bigoplus_j[\gamma_j\times\tau]^{\mathrm{Hom}(V,U_{1j}\otimes W)} \to \cdots$$

if $\tau\le\sigma$. Since $\mathrm{Hom}(V,\cdot)$ is an exact functor on the category of vector spaces, this complex is exact, and the claim follows.

The sheaves $f^*\underline{\underline{\mathrm{Hom}}}([\sigma]^V,[\tau]^W)$ and $\underline{\underline{\mathrm{Hom}}}(f^*[\sigma]^V,f^*[\tau]^W)$ are both naturally isomorphic to $f^*[\tau]^{\mathrm{Hom}(V,W)}$ for $\tau\le\sigma$ and 0 for $\tau\nleq\sigma$ and these isomorphisms are compatible with the boundary maps in the double complexes $f^*\underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{J}}^\bullet,\underline{\underline{K}}^\bullet)$ and $\underline{\underline{\mathrm{Hom}}}^\bullet(f^*\underline{\underline{J}}^\bullet,f^*\underline{\underline{K}}^\bullet)$, so we have that their associated single complexes are isomorphic. Hence

$f^* R^\bullet \underline{\underline{\mathrm{Hom}}}(\underline{\underline{J}}^\bullet, \underline{\underline{K}}^\bullet) \overset{qi}{\equiv} R^\bullet \underline{\underline{\mathrm{Hom}}}(f^* \underline{\underline{J}}^\bullet, f^* \underline{\underline{K}}^\bullet)$.

Given a cell complex X, we can always form a cellular map $f : X \to P$ where P is the cell complex consisting of a single point. Note that $D^b(P)$ can be identified with $D^b(VS) = K^b(VS)$. f is clearly a fibred map.

Theorem 3.4.14: If $f : X \to P$ is the map of X to a point and $\underline{\underline{A}}^\bullet \in D^b(X)$, then $\Gamma \underline{\underline{A}}^\bullet = f_* \underline{\underline{A}}^\bullet$ and $\Gamma_c \underline{\underline{A}}^\bullet = f_! \underline{\underline{A}}^\bullet$.

The proof is immediate from the definitions.

Corollary 3.4.15: For $\underline{\underline{A}}^\bullet \in D_f^b(X)$, $R^\bullet \Gamma_c \underline{\underline{A}}^\bullet \overset{qi}{\equiv} DR^\bullet \Gamma D \underline{\underline{A}}^\bullet$ (where, for $V^\bullet \in K_f^b(VS)$, $DV^\bullet$ is the dual chain complex).

Proof: Use theorem 3.4.14 and the fact that $R^\bullet f_! \underline{\underline{A}} \overset{qi}{\equiv} DR^\bullet f_* D \underline{\underline{A}}$.

Corollary 3.4.16: If $g : X \to Y$ is a cellular map and $\underline{\underline{A}}^\bullet \in D^b(X)$, then $\mathbb{H}^i(X, \underline{\underline{A}}^\bullet) \cong \mathbb{H}^i(Y, R^\bullet g_* \underline{\underline{A}}^\bullet)$. If g is fibred and $\underline{\underline{A}}^\bullet \in D_f^b(X)$, then $\mathbb{H}_c^i(X, \underline{\underline{A}}^\bullet) \cong \mathbb{H}_c^i(Y, R^\bullet g_! \underline{\underline{A}}^\bullet)$.

Proof: Let $f : Y \to P$ be the map to a point. Then

$$\mathbb{H}^i(X,\underline{\underline{A}}^\bullet) \cong H^i(R^\bullet \Gamma \underline{\underline{A}}^\bullet) \cong H^i(R^\bullet (fg)_* \underline{\underline{A}}^\bullet) \cong H^i(R^\bullet f_* R^\bullet g_* \underline{\underline{A}}^\bullet) \cong H^i(R^\bullet \Gamma R^\bullet g_* \underline{\underline{A}}^\bullet)$$
$$= \mathbb{H}^i(Y, R^\bullet g_* \underline{\underline{A}}^\bullet) .$$

The proof of the other statement is identical, replacing $*$ with $!$ and $R^\bullet \Gamma$ with $R^\bullet \Gamma_c$.

CHAPTER FOUR

TRIANGLES

Although $K^b(X)$ and $D^b(X)$ are not abelian categories, they each have the structure of a <u>triangulated category</u>, and from this structure, some of the machinery of abelian categories can be salvaged.

Let $f^\bullet : \underline{\underline{S}}^\bullet \to \underline{\underline{T}}^\bullet$ be a chain map between bounded complexes of sheaves, and let $M(f^\bullet)$ be the mapping cone of $f^\bullet$. Then we have a diagram of chain maps

$$\underline{\underline{S}}^\bullet \xrightarrow{f^\bullet} \underline{\underline{T}}^\bullet \hookrightarrow M(f^\bullet) \twoheadrightarrow \underline{\underline{S}}^\bullet[1]$$

which represents a diagram in $K^b(X)$.

<u>Definition</u>: A <u>triangle</u> in $K^b(X)$ is a diagram

$$\underline{\underline{A}}^\bullet \overset{u}{\to} \underline{\underline{B}}^\bullet \overset{v}{\to} \underline{\underline{C}}^\bullet \overset{w}{\to} \underline{\underline{A}}^\bullet[1]$$

which is isomorphic in $K^b(X)$ to a diagram represented by $\underline{\underline{S}}^\bullet \xrightarrow{f^\bullet} \underline{\underline{T}}^\bullet \hookrightarrow M(f^\bullet) \twoheadrightarrow \underline{\underline{S}}^\bullet[1]$ for some chain map $f^\bullet : \underline{\underline{S}}^\bullet \to \underline{\underline{T}}^\bullet$.

By isomorphic we mean that there are $K^b(X)$ - isomorphisms $\underline{\underline{A}}^\bullet \to \underline{\underline{S}}^\bullet$, $\underline{\underline{B}}^\bullet \to \underline{\underline{T}}^\bullet$, and $\underline{\underline{C}}^\bullet \to M(f^\bullet)$, commuting with the maps in the diagrams. We will frequently write a triangle as

$$\begin{tikzcd} & \underline{\underline{C}}^\bullet \arrow[dl, "{[1]}"', "w"] & \\ \underline{\underline{A}}^\bullet \arrow[rr, "u"] & & \underline{\underline{B}}^\bullet \arrow[ul, "v"'] \end{tikzcd}$$

Triangles can be thought of as analogues of short exact sequences of chain complexes in that they have associated long exact sequences of vector spaces or sheaves. For example, applying $\underline{\underline{H}}^0$ to the triangle $\underline{\underline{A}}^\bullet \overset{u}{\to} \underline{\underline{B}}^\bullet \overset{v}{\to} \underline{\underline{C}}^\bullet \overset{w}{\to} \underline{\underline{A}}^\bullet[1]$ gives a sequence of sheaves

$$\cdots \to \underline{\underline{H}}^{i-1}(\underline{\underline{C}}^\bullet) \to \underline{\underline{H}}^i\underline{\underline{A}}^\bullet \to \underline{\underline{H}}^i\underline{\underline{B}}^\bullet \to \underline{\underline{H}}^i\underline{\underline{C}}^\bullet \to \underline{\underline{H}}^{i+1}\underline{\underline{A}}^\bullet \to \cdots$$

since $\underline{\underline{H}}^0(\underline{\underline{S}}^\bullet[i]) = \underline{\underline{H}}^i(\underline{\underline{S}}^\bullet)$. But this sequence is exact, since it is isomorphic to the long exact sequence associated to the short exact sequence of chain maps

$$0 \to \underline{\underline{T}}^\bullet \to M(f^\bullet) \to \underline{\underline{S}}^\bullet[1] \to 0$$

for some chain map $f^\bullet : \underline{\underline{S}}^\bullet \to \underline{\underline{T}}^\bullet$.

Note that since $K^b(X)$ is not an abelian category, we cannot talk about short exact sequences in $K^b(X)$. However, it will be shown (theorem 4.8) that if $0 \to \underline{\underline{A}}^\bullet \xrightarrow{j^\bullet} \underline{\underline{B}}^\bullet \xrightarrow{\pi^\bullet} \underline{\underline{C}}^\bullet \to 0$ is a split exact sequence of chain maps, then the corresponding long exact sequence of sheaves can also be formed from an associated triangle $\underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet \to \underline{\underline{A}}^\bullet[1]$ in $K^b(X)$ whose first two maps are represented by the chain maps $j^\bullet$ and $\pi^\bullet$.

<u>Theorem 4.1</u>: Given two triangles $\underline{\underline{A}}^\bullet \overset{r}{\to} \underline{\underline{B}}^\bullet \overset{s}{\to} \underline{\underline{C}}^\bullet \overset{t}{\to} \underline{\underline{A}}^\bullet[1]$ and $\underline{\underline{D}}^\bullet \overset{u}{\to} \underline{\underline{E}}^\bullet \overset{v}{\to} \underline{\underline{F}}^\bullet \overset{w}{\to} \underline{\underline{D}}^\bullet[1]$, and maps $f : \underline{\underline{A}}^\bullet \to \underline{\underline{D}}^\bullet$, $g : \underline{\underline{B}}^\bullet \to \underline{\underline{E}}^\bullet$ commuting with r and u , there is a map $h : \underline{\underline{C}}^\bullet \to \underline{\underline{F}}^\bullet$ such that $wh = ft$ and $vg = hs$ (in $K^b(X)$!).

<u>Proof</u>: We can assume that the triangles are represented by diagrams of the form $\underline{\underline{A}}^\bullet \xrightarrow{r^\bullet} \underline{\underline{B}}^\bullet \to M(r^\bullet) \to \underline{\underline{A}}^\bullet[1]$ for $r^\bullet$ a chain map. Then we want to show that if the following diagram of chain maps commutes up to homotopy, there is a chain map $h^\bullet : M(r^\bullet) \to M(u^\bullet)$ for which the diagram still commutes up to homotopy:

$$\begin{tikzcd}
\underline{\underline{A}}^\bullet \arrow[r, "r^\bullet"] \arrow[d, "f^\bullet"] & \underline{\underline{B}}^\bullet \arrow[r] \arrow[d, "g^\bullet"] & M(r^\bullet) \arrow[r] & \underline{\underline{A}}^\bullet[1] \arrow[d, "f^\bullet"] \\
\underline{\underline{D}}^\bullet \arrow[r, "u^\bullet"] & \underline{\underline{E}}^\bullet \arrow[r] & M(u^\bullet) \arrow[r] & \underline{\underline{D}}^\bullet[1]
\end{tikzcd}$$

Let $g^{\bullet}r^{\bullet} - u^{\bullet}f^{\bullet}$ be given by a homotopy $T^{\bullet}$, i.e., $g^i r^i - u^i f^i = T^{i+1}\partial^i + \partial^{i-1}T^i$, where $T^i : \underline{\underline{A}}^i \to \underline{\underline{E}}^{i-1}$. Then it is easily verified that the map

$$\begin{tikzcd}
\underline{\underline{A}}^{i+1} \arrow[r, "f^{i+1}"] \arrow[ddr, "(-1)^{i+1}T^{i+1}"] & \underline{\underline{D}}^{i+1} \\
\oplus & \oplus \\
\underline{\underline{B}}^{i} \arrow[r, "g^{i}"] & \underline{\underline{E}}^{i}
\end{tikzcd}$$

defines a chain map, and commutes with the above diagram.

By a <u>map</u> between trianges $\underline{\underline{A}}^{\bullet} \to \underline{\underline{B}}^{\bullet} \to \underline{\underline{C}}^{\bullet} \to \underline{\underline{A}}^{\bullet}[1]$ and $\underline{\underline{D}}^{\bullet} \to \underline{\underline{E}}^{\bullet} \to \underline{\underline{F}}^{\bullet} \to \underline{\underline{D}}^{\bullet}[1]$, we mean a collection of $K^b(X)$ maps $\underline{\underline{A}}^{\bullet} \to \underline{\underline{D}}^{\bullet}$, $\underline{\underline{B}}^{\bullet} \to \underline{\underline{E}}^{\bullet}$, $\underline{\underline{C}}^{\bullet} \to \underline{\underline{F}}^{\bullet}$ such that

$$\begin{tikzcd}
 & \underline{\underline{C}}^{\bullet} \arrow[dl, "{[1]}"'] \arrow[rrr] & & & \underline{\underline{E}}^{\bullet} \arrow[dl, "{[1]}"'] & \\
\underline{\underline{A}}^{\bullet} \arrow[rr] \arrow[rrr, bend left=10] & & \underline{\underline{B}}^{\bullet} \arrow[ul] \arrow[rrr, bend right=10] & \underline{\underline{D}}^{\bullet} \arrow[rr] & & \underline{\underline{E}}^{\bullet} \arrow[ul]
\end{tikzcd}$$

commutes.

<u>Theorem 4.2</u>: Given a map between two triangles, if the two maps between the bottom objects in the triangle are isomorphisms, then the

map between the top objects is an isomorphism.

Proof: We may assume that the triangles are represented by diagrams of chain maps of the form $\underline{\underline{A}}^\bullet \xrightarrow{u^\bullet} \underline{\underline{B}}^\bullet \to M(u^\bullet) \to \underline{\underline{A}}^\bullet[1]$. We will first assume that the two isomorphisms are identity maps, i.e., given any diagram of chain maps

$$\begin{tikzcd}
 & & M(u^\bullet) \arrow[dd, "h^\bullet"] \arrow[dr, "\pi^\bullet"] & \\
\underline{\underline{A}}^\bullet \arrow[r, "u^\bullet"] & \underline{\underline{B}}^\bullet \arrow[ur, "j^\bullet"] \arrow[dr, "j^\bullet"'] & & \underline{\underline{A}}^\bullet[1] \\
 & & M(u^\bullet) \arrow[ur, "\pi^\bullet"'] &
\end{tikzcd}$$

commuting up to homotopy, $h^\bullet$ must have a homotopy inverse.

Since $h^\bullet j^\bullet = j^\bullet$ up to homotopy, there is a homotopy $T^i : \underline{\underline{B}}^i \to \underline{\underline{A}}^i \oplus \underline{\underline{B}}^{i-1}$ such that $h^i j^i = \partial^{i-1} T^i + T^{i+1}\partial^i$. But T^i can be factored as $\underline{\underline{B}}^i \xrightarrow{j^i} \underline{\underline{A}}^{i+1} \oplus \underline{\underline{B}}^i \xrightarrow{\tilde{T}^i} \underline{\underline{A}}^i \oplus \underline{\underline{B}}^{i-1}$, so a map homotopic to zero can be added to h to make $h^\bullet j^\bullet = j^\bullet$.

We also have a homotopy $s^i : \underline{\underline{A}}^{i+1} \oplus \underline{\underline{B}}^i \to \underline{\underline{A}}^i$ such that $\pi^i h^i - \pi^i = \partial^i s^i + s^{i+1}\partial^i$. s^i can be factored as $\underline{\underline{A}}^{i+1} \oplus \underline{\underline{B}}^i \xrightarrow{\tilde{s}^i} \underline{\underline{A}}^i \oplus \underline{\underline{B}}^{i-1} \xrightarrow{\pi^{i-1}} \underline{\underline{A}}^i$ where $\tilde{s}^i$ is the 0-map into the summand $\underline{\underline{B}}^{i-1}$. Then $\pi^i h^i - \pi^i = \partial^i s^i - s^{i+1}\partial^i = \partial^i \pi^{i-1} \tilde{s}^i + \pi^i \tilde{s}^{i+1} \partial^i = \pi^i(\partial^i \tilde{s}^i + \tilde{s}^{i+1}\partial^i)$. $h^i j^i = j^i$ implies that h^i is 0 between the components $\underline{\underline{B}}^i$ and $\underline{\underline{A}}^{i+1}$, so $\pi^i h^i - \pi^i$ is 0 on $\underline{\underline{B}}^i$.

Then $\pi^{i+1}(\partial^i\tilde{s}^i + \tilde{s}^{i+1}\partial^i)$ is 0 on $\underline{\underline{B}}^i$, i.e., $\partial^i\tilde{s}^i + \tilde{s}^{i+1}\partial^i$ is 0 on the component $\underline{\underline{B}}^i \to \underline{\underline{A}}^{i+1}$, and hence is 0 on $\underline{\underline{B}}^i$. Then we have $\pi^i(h^i - \partial^i\tilde{s}^i - \tilde{s}^{i+1}\partial^i) = \pi^i$, and $(h^i - \partial^i\tilde{s}^i - \tilde{s}^{i+1}\partial^i)j^i = j^i$.

Therefore, by changing $h^\bullet$ in the original diagram to a map that represents the same map in $K^b(X)$, we can assume that h^i is 0 between the components $\underline{\underline{B}}^i$ and $\underline{\underline{A}}^{i+1}$, and that the diagram commutes. It is clear that h^i must be of the form

$$\begin{tikzcd} \underline{\underline{A}}^{i+1} \arrow[r, "id"] \arrow[ddr, "r^{i+1}"] & \underline{\underline{A}}^{i+1} \\ \oplus & \oplus \\ \underline{\underline{B}}^i \arrow[r, "id"'] & \underline{\underline{B}}^i \end{tikzcd}$$

Since this is a chain map, if we define $\tilde{h}^i$ to be the same map but with r^{i+1} replaced with $-r^{i+1}$, this must also be a chain map, and is easily seen to be an inverse of $h^\bullet$, so $h^\bullet$ is an isomorphism.

Now assume we have a diagram of chain maps

$$\begin{tikzcd} & & M(v^\bullet) \arrow[dd, "{[1]}"] & \\ M(u^\bullet) \arrow[urr, "h^\bullet"] \arrow[ddd, "{[1]}"'] & & & \\ & & \underline{\underline{C}}^\bullet \arrow[r, "v^\bullet"] & \underline{\underline{D}}^\bullet \arrow[uul] \\ \underline{\underline{A}}^\bullet \arrow[r, "u^\bullet"] \arrow[urr, "f^\bullet"] & \underline{\underline{B}}^\bullet \arrow[uuul] \arrow[urr, "g^\bullet"'] & & \end{tikzcd}$$

that commutes up to homotopy, where $f^\bullet$ and $g^\bullet$ have homotopy inverses, $\tilde{f}^\bullet$ and $\tilde{g}^\bullet$. By theorem 4.1, there is a diagram

$$\begin{tikzcd}
 & & & M(v^\bullet) \arrow[dl, "{[1]}"'] & \\
 & M(u^\bullet) \arrow[urr, "k^\bullet"] \arrow[dl, "{[1]}"'] & \underline{\underline{C}}^\bullet \arrow[rr, "v^\bullet"] & & \underline{\underline{D}}^\bullet \arrow[ul] \\
\underline{\underline{A}}^\bullet \arrow[r, "u"] \arrow[urr, "\tilde{f}^\bullet"] & \underline{\underline{B}}^\bullet \arrow[u] \arrow[urrr, "\tilde{g}^\bullet"'] & & &
\end{tikzcd}$$

commuting up to homotopy, which gives us

$$\begin{tikzcd}
M(v^\bullet) \arrow[r, "k^\bullet h^\bullet"] \arrow[d, "{[1]}"'] & M(v^\bullet) \arrow[dl, "{[1]}"'] \\
\underline{\underline{C}}^\bullet \arrow[r, "v^\bullet"] & \underline{\underline{D}}^\bullet \arrow[ul] \arrow[u]
\end{tikzcd}$$

Then $k^\bullet h^\bullet$ is an isomorphism in $K^b(X)$, and similarly so is $h^\bullet k^\bullet$, so $h^\bullet$ is an isomorphism in $K^b(X)$.

Corollary 4.3: Given $f : \underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet$ in $K^b(X)$, there is a unique (up to $K^b(X)$-isomorphism) triangle of the form

$$\begin{tikzcd}
 & \underline{\underline{A}}^\bullet \arrow[dl, "h", "{[1]}"'] & \\
\underline{\underline{B}}^\bullet \arrow[rr, "f"] & & \underline{\underline{C}}^\bullet \arrow[ul, "g"']
\end{tikzcd}$$

Proof: This follows immediately from theorem 4.1 and the first case considered in the proof of theorem 4.2.

<u>Remark</u>: This shows that the mapping cone of a $K^b(X)$ map is well-defined up to $K^b(X)$-isomorphism. It can be shown by explicit construction, though, that for f a map in $K^b(X)$, $M(f)$ is well-defined up to isomorphism in the category of chain complexes and chain maps, and not just up to $K^b(X)$-isomorphism (where $M(f) = M(f^\bullet)$ for some representative $f^\bullet$ of f).

<u>Theorem 4.4</u>:

$$\begin{tikzcd} & 0 \arrow[dl, "{[1]}"'] & \\ \underline{\underline{A}}^\bullet \arrow[rr, "id"] & & \underline{\underline{A}}^\bullet \arrow[ul] \end{tikzcd}$$

is a triangle.

<u>Proof</u>: We need to show that the identity map on $M(id)$ is homotopic to zero. It is easily verified that $T^i : \underline{\underline{A}}^{i+1} \oplus \underline{\underline{A}}^i \to \underline{\underline{A}}^i \oplus \underline{\underline{A}}^{i-1}$ which maps $\underline{\underline{A}}^i \to \underline{\underline{A}}^i$ by $(-1)^i id$ and is zero between all other summands gives the desired homotopy.

<u>Theorem 4.5</u>: $\underline{\underline{A}}^\bullet \overset{u}{\to} \underline{\underline{B}}^\bullet \overset{v}{\to} \underline{\underline{C}}^\bullet \overset{w}{\to} \underline{\underline{A}}^\bullet[1]$, is a triangle if and only if $\underline{\underline{B}}^\bullet \overset{v}{\to} \underline{\underline{C}}^\bullet \overset{w}{\to} \underline{\underline{A}}^\bullet[1] \xrightarrow{-u[1]} \underline{\underline{B}}^\bullet[1]$ is a triangle.

<u>Proof</u>: Given a chain map $u^\bullet : \underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet$, we want to show that the diagrams of chain maps

$$\begin{tikzcd} & \underline{\underline{A}}^\bullet[1] \arrow[dl, "{[1]}"', "{-u^\bullet[1]}"] & \\ \underline{\underline{B}}^\bullet \arrow[rr, "v^\bullet"] & & M(u^\bullet) \arrow[ul, "w^\bullet"'] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & M(v^\bullet) \arrow[dl, "{[1]}"', "\pi^\bullet"] & \\ \underline{\underline{B}}^\bullet \arrow[rr, "v^\bullet"] & & M(u^\bullet) \arrow[ul, "j^\bullet"'] \end{tikzcd}$$

represent isomorphic diagrams in $K^b(X)$, where $v^\bullet, w^\bullet, \pi^\bullet$, and $j^\bullet$ are the natural maps.

$M(v^\bullet)$ is the complex with $\underline{\underline{B}}^{i+1} \oplus \underline{\underline{A}}^{i+1} \oplus \underline{\underline{B}}^i$ in degree i , and boundary maps

$$\begin{tikzcd}
\underline{\underline{B}}^{i+1} \arrow[d, "\partial^{i+1}"'] \arrow[drrrr, "(-1)^{i+1} id" near start] & \oplus & \underline{\underline{A}}^{i+1} \arrow[d, "\partial^{i+1}"'] \arrow[drr, "(-1)^{i+1} u^{i+1}"] & \oplus & \underline{\underline{B}}^{i} \arrow[d, "\partial^{i}"] \\
\underline{\underline{B}}^{i+2} & \oplus & \underline{\underline{A}}^{i+2} & \oplus & \underline{\underline{B}}^{i+1}
\end{tikzcd}$$

We can define a chain map $h^\bullet : \underline{\underline{A}}^\bullet[1] \to M(v^\bullet)$ by letting $h^\bullet : \underline{\underline{A}}^{i+1} \to \underline{\underline{B}}^{i+1} \oplus \underline{\underline{A}}^{i+1} \to \underline{\underline{B}}^i$ be $(-u^{i+1}, id, 0)$. This gives us a diagram

$$\begin{tikzcd}
 & & \underline{\underline{A}}^\bullet[1] \arrow[dd, "h^\bullet"] \arrow[dr, "-u^\bullet[1]"] & \\
\underline{\underline{B}}^\bullet \arrow[r, "v^\bullet"] & M(u^\bullet) \arrow[ur, "w^\bullet"] \arrow[dr, "j^\bullet"'] & & \underline{\underline{B}}^\bullet[1] \\
 & & M(v^\bullet) \arrow[ur, "\pi^\bullet"'] &
\end{tikzcd}$$

It is easily checked that $\pi^\bullet h^\bullet = -u^\bullet[1]$.

$h^i w^i : \underline{\underline{A}}^{i+1} \oplus \underline{\underline{B}}^i \to \underline{\underline{B}}^{i+1} \oplus \underline{\underline{A}}^{i+1} \oplus \underline{\underline{B}}^i$ is 0 restricted to $\underline{\underline{B}}^\bullet$ and

$(-u^{i+1}, id, 0)$ restricted to $\underline{\underline{A}}^{i+1}$. Then $j^i - h^i w^i$ is the map

$$\begin{tikzcd} & \underline{\underline{A}}^{i+1} \oplus \underline{\underline{B}}^i \arrow[dl, "u^{i+1}"'] \arrow[dr, "id"] & \\ \underline{\underline{B}}^{i+1} \oplus & \underline{\underline{A}}^{i+1} & \oplus \underline{\underline{B}}^i \end{tikzcd},$$

which is homotopic to zero by the homotopy $T^i : \underline{\underline{A}}^{i+1} \oplus \underline{\underline{B}}^i \to \underline{\underline{B}}^i \oplus \underline{\underline{A}}^i \oplus \underline{\underline{B}}^{i-1}$ which is $(-1)^i id$ on $\underline{\underline{B}}^i \to \underline{\underline{B}}^i$ and 0 between all other components. Hence the diagram commutes up to homotopy.

It remains to show that $h^\bullet$ is an isomorphism in $K^b(X)$. $M(v^\bullet)$ is isomorphic to the mapping cone of $\underline{\underline{A}}^\bullet \to M(\underline{\underline{B}}^\bullet \xrightarrow{id} \underline{\underline{B}}^\bullet)$, so we have a triangle represented by

$$\begin{tikzcd} & M(v^\bullet) \arrow[dl, "{[1]}"'] & \\ \underline{\underline{A}}^\bullet \arrow[rr] & & M(id_{\underline{\underline{B}}^\bullet}) \arrow[ul] \end{tikzcd}.$$

By theorem 4.4, $M(id_{\underline{\underline{B}}^\bullet})$ is isomorphic to 0 in $K^b(X)$, so

$$\begin{tikzcd} & M(v^\bullet) \arrow[dl, "{[1]}"'] & \\ \underline{\underline{A}}^\bullet \arrow[rr] & & 0 \arrow[ul] \end{tikzcd}$$

represents a triangle. We also have a triangle

$$\begin{tikzcd} & \underline{\underline{A}}^\bullet[1] \arrow[dl, "{[1]}"', "id"] & \\ \underline{\underline{A}}^\bullet \arrow[rr] & & 0 \arrow[ul] \end{tikzcd}$$

and this triangle maps into the previous one by $id : \underline{\underline{A}}^\bullet \to \underline{\underline{A}}^\bullet$ and $h^\bullet : A^\bullet[1] \to M(v^\bullet)$. By theorem 4.2, then, $h^\bullet$ is an isomorphism in $K^b(X)$.

The converse statement, that $\underline{\underline{B}}^\bullet \overset{v}{\to} \underline{\underline{C}}^\bullet \overset{w}{\to} \underline{\underline{A}}^\bullet[1] \xrightarrow{-u[1]} \underline{\underline{B}}^\bullet[1]$ being a triangle implies that $\underline{\underline{A}}^\bullet \overset{u}{\to} \underline{\underline{B}}^\bullet \overset{v}{\to} \underline{\underline{C}}^\bullet \overset{w}{\to} \underline{\underline{A}}^\bullet[1]$ is a triangle, follows now, by applying the first statement five times to acquire the triangle

$$\underline{\underline{A}}^\bullet[2] \xrightarrow{u[2]} \underline{\underline{B}}^\bullet[2] \xrightarrow{v[2]} \underline{\underline{C}}^\bullet[2] \xrightarrow{w[2]} \underline{\underline{A}}^\bullet[3] .$$

Since $M(f^\bullet[2]) = M(f^\bullet)[2]$, we have the desired triangle.

Corollary 4.6: The composition of two successive maps in a triangle is zero.

Proof: In $\underline{\underline{A}}^\bullet \xrightarrow{u} \underline{\underline{B}}^\bullet \xrightarrow{v} M(u^\bullet) \xrightarrow{w} \underline{\underline{A}}^\bullet[1]$, wv is clearly zero. The corollary follows then from theorem 4.5.

Corollary 4.7:

$$\begin{tikzcd} & 0 \arrow[dl, "{[1]}"'] & \\ \underline{\underline{A}}^\bullet \arrow[rr, "f"] & & \underline{\underline{B}}^\bullet \arrow[ul] \end{tikzcd}$$

is a triangle if and only if f is an isomorphism.

Proof: There is an obvious map from

$$\begin{tikzcd} & 0 \arrow[dl, "{[1]}"'] & \\ \underline{\underline{A}}^\bullet \arrow[rr, "id"] & & \underline{\underline{A}}^\bullet \arrow[ul] \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} & 0 \arrow[dl, "{[1]}"'] & \\ \underline{\underline{A}}^\bullet \arrow[rr, "f"] & & \underline{\underline{B}}^\bullet \arrow[ul] \end{tikzcd}$$

If f is an isomorphism, then the diagrams are isomorphic, so the latter diagram is a triangle by theorem 4.4. If the diagrams are both triangles, then since two of the three maps between them are isomorphisms, by theorems 4.2 and 4.5, the third is as well, i.e., f is an isomorphism.

Theorem 4.8: If $0 \to \underline{\underline{A}}^\bullet \xrightarrow{j^\bullet} \underline{\underline{B}}^\bullet \xrightarrow{\pi^\bullet} \underline{\underline{C}}^\bullet \to 0$ is a split exact sequence of bounded chain complexes and chain maps, then there is a triangle

$$\begin{tikzcd} & \underline{\underline{C}}^\bullet \arrow[dl, "t", "{[1]}"'] & \\ \underline{\underline{A}}^\bullet \arrow[rr, "j"] & & \underline{\underline{B}}^\bullet \arrow[ul, "\pi"'] \end{tikzcd}$$

with j and π represented by $j^\bullet$ and $\pi^\bullet$, and such that the long exact sequence of sheaves associated to this triangle is the same as the long exact sequence associated to $0 \to \underline{\underline{A}}^\bullet \xrightarrow{j^\bullet} \underline{\underline{B}}^\bullet \xrightarrow{\pi^\bullet} \underline{\underline{C}}^\bullet \to 0$.

Proof: Let $s^i : \underline{\underline{C}}^i \to \underline{\underline{B}}^i$ be sheaf maps splitting the sequence, i.e., $\pi^i s^i = \mathrm{id}$ for each i ($s^\bullet$ is not in general a chain map). Then $(-1)^i(\partial^i s^i - s^{i+1}\partial^i) : \underline{\underline{C}}^i \to \underline{\underline{B}}^{i+1}$ defines a chain map, and $\pi^{i+1}(\partial^i s^i - s^{i+1}\partial^i) = \partial^i \pi^i s^i - \partial^i = \partial^i - \partial^i = 0$, so $(-1)^i(\partial^i s^i - s^{i+1}\partial^i)$ maps into $\underline{\underline{\ker}}(\pi^{i+1}) = \underline{\underline{\mathrm{Im}}}(j^{i+1})$, hence there is a chain map $t^\bullet : \underline{\underline{C}}^\bullet \to \underline{\underline{A}}^\bullet[1]$ with $j^{i+1}t^i = (-1)^i(\partial^i s^i - s^{i+1}\partial^i)$.

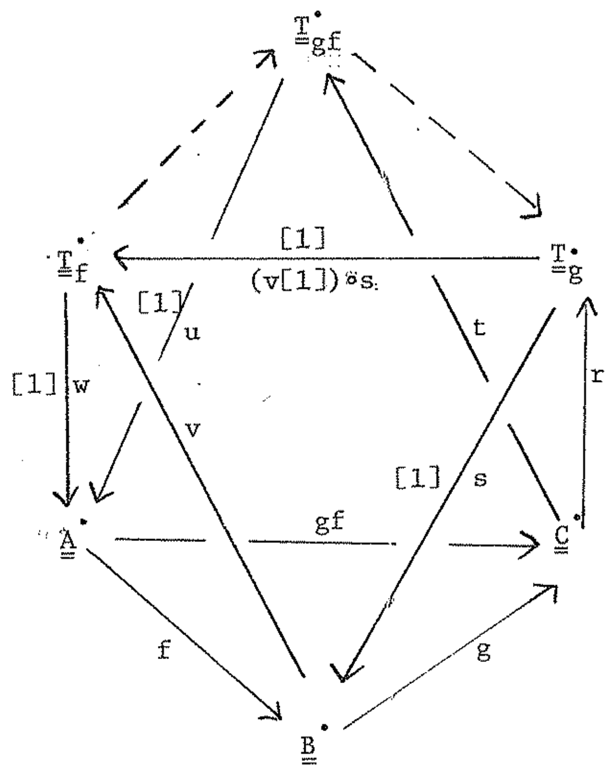
Let $r^i : \underline{\underline{C}}^{i+1} \oplus \underline{\underline{A}}^{i+1} \to \underline{\underline{B}}^{i+1}$ be s^{i+1} on $\underline{\underline{C}}^{i+1}$ and j^{i+1} on $\underline{\underline{A}}^{i+1}$. Then r^i is an isomorphism, and $r^\bullet : M(t^\bullet) \to \underline{\underline{B}}^\bullet[1]$ can easily be verified to be a chain map. We then have a commutative diagram

$$\begin{tikzcd}
 & & M(t^\bullet) \arrow[dd, "r^\bullet"] \arrow[dr] & \\
\underline{\underline{C}}^\bullet \arrow[r, "t^\bullet"] & \underline{\underline{A}}^\bullet[1] \arrow[ur, hook] \arrow[dr, "j^\bullet[1]"'] & & \underline{\underline{C}}^\bullet[1] \\
 & & \underline{\underline{B}}^\bullet[1] \arrow[ur, "\pi^\bullet[1]"'] &
\end{tikzcd}$$

which shows that $\underline{\underline{C}}^\bullet \xrightarrow{t^\bullet} \underline{\underline{A}}^\bullet[1] \xrightarrow{j^\bullet[1]} \underline{\underline{B}}^\bullet[1] \xrightarrow{\pi^\bullet[1]} \underline{\underline{C}}^\bullet[1]$ represents a triangle. By theorem 4.5, then, $\underline{\underline{A}}^\bullet \xrightarrow{-j^\bullet} \underline{\underline{B}}^\bullet \xrightarrow{-\pi^\bullet} \underline{\underline{C}}^\bullet \xrightarrow{t^\bullet} \underline{\underline{A}}^\bullet[1]$ is a triangle in $K^b(X)$, and hence so is $\underline{\underline{A}}^\bullet \xrightarrow{j^\bullet} \underline{\underline{B}}^\bullet \xrightarrow{\pi^\bullet} \underline{\underline{C}}^\bullet \xrightarrow{t^\bullet} \underline{\underline{A}}^\bullet[1]$ (map $\underline{\underline{B}}^\bullet$ to $\underline{\underline{B}}^\bullet$ by $-\mathrm{id}$). The fact that this gives out the same long exact sequence as $0 \to \underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet \to 0$ is clear.

<u>Remark</u>: The condition that the short exact sequence is split is necessary. However, if either $\underline{\underline{A}}^\bullet$ is an injective complex or $\underline{\underline{C}}^\bullet$ is a projective complex, then the sequence is always split.

Let $\underline{\underline{A}}^\bullet \overset{f}{\to} \underline{\underline{B}}^\bullet \overset{g}{\to} \underline{\underline{C}}^\bullet$ be maps in $K^b(X)$. Then one can form the diagram in $K^b(X)$ of solid lines below, where $\underline{\underline{T}}_f^\bullet$, $\underline{\underline{T}}_g^\bullet$, and $T_{gf}^\bullet$ are the third complexes in triangles over f, g, and gf, respectively:



<u>Theorem 4.9</u>: In the above diagram, there exist maps $\alpha : \underline{\underline{T}}^{\bullet}_{f} \to \underline{\underline{T}}^{\bullet}_{gf}$ and $\beta : \underline{\underline{T}}^{\bullet}_{gf} \to \underline{\underline{T}}^{\bullet}_{g}$ such that

$$\begin{tikzcd}
 & \underline{\underline{T}}^{\bullet}_{g} \arrow[dl, "{[1]}"', "{(v[1])\circ s}"] & \\
\underline{\underline{T}}^{\bullet}_{f} \arrow[rr, "\alpha"'] & & \underline{\underline{T}}^{\bullet}_{gf} \arrow[ul, "\beta"']
\end{tikzcd}$$

is a triangle, and the diagram commutes.

Proof: $\underline{\underline{T}}^\bullet_f$, $\underline{\underline{T}}^\bullet_g$, and $\underline{\underline{T}}^\bullet_{gf}$ can be taken to be the mapping cones of representatives $f^\bullet$, $g^\bullet$, and $g^\bullet f^\bullet$ of f , g , and gf , respectively. We can then define α and β to be given by chain maps $\alpha^\bullet$ and $\beta^\bullet$ which in degree i are

$$\begin{array}{ccc} \underline{\underline{A}}^{i+1} & \oplus & \underline{\underline{B}}^{i} \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle g^i} \\ \underline{\underline{A}}^{i+1} & \oplus & \underline{\underline{C}}^{i} \end{array} \quad \text{and} \quad \begin{array}{ccc} \underline{\underline{A}}^{i+1} & \oplus & \underline{\underline{C}}^{i} \\ \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle id} \\ \underline{\underline{B}}^{i+1} & \oplus & \underline{\underline{C}}^{i} \end{array} \quad \text{, respectively.}$$

It is clear that $u\alpha = w$, $\beta t = r$, $fu = s\beta$, and $\alpha v = tg$.

We must show that

$$\begin{tikzcd} & \underline{\underline{T}}^\bullet_g \arrow[dl, "(v[1])s[1]"'] & \\ \underline{\underline{T}}^\bullet_f \arrow[rr, "\alpha"] & & \underline{\underline{T}}^\bullet_{gf} \arrow[ul, "\beta"'] \end{tikzcd}$$ is a triangle.

$M(\alpha^\bullet)$ is the complex

$$\begin{tikzcd}[column sep=small]
(\underline{\underline{A}}^{i+2} \arrow[d, "\partial^{i+2}"'] \arrow[drr, "(-1)^{i+1} f^{i+2} \partial^{i+1}"] \arrow[drrrr, "(-1)^{i+1} id"] & \oplus & \underline{\underline{B}}^{i+1}) \arrow[d, "\partial^{i+1}"] \arrow[drrrr, "(-1)^{i+1} g^{i+1}"] & \oplus & (\underline{\underline{A}}^{i+1} \arrow[d, "\partial^{i+1}"] \arrow[drr, "(-1)^{i+1} g^{i+1} f^{i+1}"] & \oplus & \underline{\underline{C}}^{i}) \arrow[d, "\partial^{i}"] & \text{degree } i \\
(\underline{\underline{A}}^{i+3} & \oplus & \underline{\underline{B}}^{i+2}) & \oplus & (\underline{\underline{A}}^{i+2} & \oplus & \underline{\underline{C}}^{i+1}) & \text{degree } i+1
\end{tikzcd}$$

There is a chain map $\gamma^{\bullet} : \underline{\underline{T}}^{\bullet}_{g} \to M(\alpha^{\bullet})$ where $\gamma^{i} : \underline{\underline{B}}^{i+1} \oplus \underline{\underline{C}}^{i} \to \underline{\underline{A}}^{i+2} \oplus \underline{\underline{B}}^{i+1} \oplus \underline{\underline{A}}^{i+1} \oplus \underline{\underline{C}}^{i}$ is the identity on components $\underline{\underline{B}}^{i+1} \to \underline{\underline{B}}^{i+1}$ and $\underline{\underline{C}}^{i} \to \underline{\underline{C}}^{i}$ and zero between other components. $(v^{\bullet}[1])s^{\bullet}$ is the map $\underline{\underline{B}}^{i+1} \oplus \underline{\underline{C}}^{i} \to \underline{\underline{A}}^{i+2} \oplus \underline{\underline{B}}^{i+1}$ on degree i that is the identity on components $\underline{\underline{B}}^{i+1} \to \underline{\underline{B}}^{i+1}$ and zero between other components, and it is easily checked that $(v^{\bullet}[1])s^{\bullet} = \lambda^{\bullet}\gamma^{\bullet}$ where $\lambda^{\bullet}$ represents the map λ in the triangle

$$\begin{tikzcd}
 & M(\alpha^{\bullet}) \arrow[dl, "{[1]}"', "\lambda"] & \\
\underline{\underline{T}}^{\bullet}_{f} \arrow[rr, "\alpha"] & & \underline{\underline{T}}^{\bullet}_{gf} \arrow[ul, "\mu"']
\end{tikzcd} .$$

The map $\gamma^{i}\beta^{i} - \mu^{i} : \underline{\underline{A}}^{i+1} \oplus \underline{\underline{C}}^{i} \to \underline{\underline{A}}^{i+2} \oplus \underline{\underline{B}}^{i+1} \oplus \underline{\underline{A}}^{i+1} \oplus \underline{\underline{C}}^{i}$ is zero on $\underline{\underline{C}}^{i}$ and $(0, f^{i+1}, -id, 0)$ on $\underline{\underline{A}}^{i+1}$, and the homotopy $T^{i} : \underline{\underline{A}}^{i+1} \oplus \underline{\underline{C}}^{i} \to \underline{\underline{A}}^{i+1} \oplus \underline{\underline{B}}^{i} \oplus \underline{\underline{A}}^{i} \oplus \underline{\underline{C}}^{i-1}$ given by $((-1)^{i+1}, id, 0, 0, 0)$ on $\underline{\underline{A}}^{i+1}$ and 0 on $\underline{\underline{C}}^{i}$ shows that this map is homotopic to zero. Then $\gamma\beta = \mu$ in $K^{b}(X)$, so we have a $K^{b}(X)$-commutative diagram,

$$\begin{tikzcd}
 & & \underline{\underline{T}}_g^\bullet \arrow[rr, "(v[1])s"] \arrow[d, "\gamma"] & & \underline{\underline{T}}_f^\bullet[1] \\
\underline{\underline{T}}_f^\bullet \arrow[r, "\alpha"] & \underline{\underline{T}}_{gf}^\bullet \arrow[ur, "\beta"] \arrow[r, "\mu"'] & M(\alpha^\bullet) \arrow[urr, "\lambda"'] & &
\end{tikzcd}$$

It remains only to show that γ is an isomorphism. Note that $M(\alpha^\bullet)$ can be viewed as being the mapping cone of the map $\ell^\bullet : M(-\mathrm{id} : \underline{\underline{A}}^\bullet \to \underline{\underline{A}}^\bullet) \to M(g^\bullet)$ where $\ell^i : \underline{\underline{A}}^{i+1} \oplus \underline{\underline{A}}^i \to \underline{\underline{B}}^{i+1} \oplus \underline{\underline{C}}^i$ is $(-f^{i+1}) \oplus (g^i f^i)$. $-\mathrm{id}$ is an isomorphism in $K^b(X)$ so by Corollaries 4.3 and 4.7, $M(-\mathrm{id}_{\underline{\underline{A}}^\bullet})$ is isomorphic to 0 in $K^b(X)$. Then we have a triangle

$$\begin{tikzcd}
M(\alpha^\bullet) = M(\ell^\bullet) \arrow[d, "{[1]}"'] & \\
0 \arrow[r] & \underline{\underline{T}}_g^\bullet = M(g^\bullet) \arrow[ul]
\end{tikzcd}$$

, and it can be checked that the non-zero map is γ . Then by theorem 4.5 and corollary 4.7, γ is an isomorphism.

All constructions and theorems in this chapter can be made in $K^b(VS)$ instead of $K^b(X)$--we merely take X to be the complex consisting of a single point. To motivate theorem 4.9 in $K^b(VS)$, let $A \subseteq B$ be topological spaces. We have a short exact sequence $0 \to C_\bullet(A;\mathbb{Q}) \to C_\bullet(B;\mathbb{Q}) \to C_\bullet(B,A;\mathbb{Q}) \to 0$ of singular chain groups which is split, hence we have a triangle

$$\begin{tikzcd} & C_\bullet(B,A;\mathbb{Q}) \arrow[dl, "{[1]}"'] & \\ C_\bullet(A;\mathbb{Q}) \arrow[rr] & & C_\bullet(B;\mathbb{Q}) \arrow[ul] \end{tikzcd} \quad .$$

Now let $A \subseteq B \subseteq C$ be a triple of spaces, giving us maps $C_\bullet(A;\mathbb{Q}) \to C_\bullet(B;\mathbb{Q}) \to C_\bullet(C;\mathbb{Q})$. If the construction of theorem 4.9 is applied to these, we get a diagram relating the triangles which give the long exact sequences for the pairs (B,A), (C,A) and (C,B), to the triangle which gives the long exact sequence for the triple. Theorem 4.9 in this setting, then, shows that the existence of the long exact sequence for a triple follows from the existence of the long exact sequence for a pair.

There is a general definition for a triangulated category that $K^b(X)$ satisfies. An additive category $\mathcal{O}$ with an automorphism $T : \mathcal{O} \to \mathcal{O}$ (in our case, $T(\underline{\underline{A}}^\bullet) = \underline{\underline{A}}^\bullet[1]$) and a collection of diagrams (called <u>triangles</u>)

$$\begin{tikzcd} & Z \arrow[dl, "T/w"'] & \\ X \arrow[rr, "u"] & & Y \arrow[ul, "v"'] \end{tikzcd}$$

where $w : Z \to T(X)$, is a <u>triangulated category</u> if every map $X \xrightarrow{u} Y$ is the bottom of some triangle, every diagram isomorphic to a triangle is a triangle, and theorems 4.1, 4.4, 4.5, and 4.9 hold. Theorem 4.9 is referred to as the octahedral axiom. As will be shown, $D^b(X)$, as well as $K_f^b(X)$ and $D_f^b(X)$, are triangulated categories.

<u>Definition</u>: A diagram
$$\begin{tikzcd} & \underline{\underline{C}}^\bullet \arrow[dl, "{[1]}"', "w"] & \\ \underline{\underline{A}}^\bullet \arrow[rr, "u"] & & \underline{\underline{B}}^\bullet \arrow[ul, "v"'] \end{tikzcd}$$
in $D^b(X)$ is a <u>triangle</u> if there is a triangle
$$\begin{tikzcd} & \underline{\underline{F}}^\bullet \arrow[dl, "{[1]}"', "t"] & \\ \underline{\underline{D}}^\bullet \arrow[rr, "r"'] & & \underline{\underline{E}}^\bullet \arrow[ul, "s"'] \end{tikzcd}$$
in $K^b(X)$ and (where r,s,t are considered as representing maps in $D^b(X)$) a $D^b(X)$-commutative diagram
$$\begin{tikzcd} & & & & \underline{\underline{F}}^\bullet \arrow[dl, "{[1]}"', "t"] & \\ & \underline{\underline{C}}^\bullet \arrow[dl, "{[1]}"', "w"] \arrow[urrr, "\gamma"] & & \underline{\underline{D}}^\bullet \arrow[rr, "r"] & & \underline{\underline{E}}^\bullet \arrow[ul, "s"'] \\ \underline{\underline{A}}^\bullet \arrow[rr, "u"'] \arrow[urrr, "\alpha"] & & \underline{\underline{B}}^\bullet \arrow[ul, "v"] \arrow[urrr, "\beta"'] & & & \end{tikzcd}$$
such that α, β, and γ are isomorphisms in $D^b(X)$.

Every triangle in $D^b(X)$ is clearly isomorphic to one represented by a diagram of the form
$$\begin{tikzcd} & M(u^\bullet) \arrow[dl, "{[1]}"'] & \\ \underline{\underline{A}}^\bullet \arrow[rr, "u^\bullet"] & & \underline{\underline{B}}^\bullet \arrow[ul] \end{tikzcd}$$
for a chain map $u^\bullet$. Also, by taking injective resolutions
$$\begin{tikzcd} \underline{\underline{A}}^\bullet \arrow[r, "qi"] \arrow[d, "u"'] & \underline{\underline{I}}^\bullet \arrow[d] \\ \underline{\underline{B}}^\bullet \arrow[r, "qi"] & \underline{\underline{J}}^\bullet \end{tikzcd}$$
and noting that the mapping cone of a map between injective complexes is an injective com-

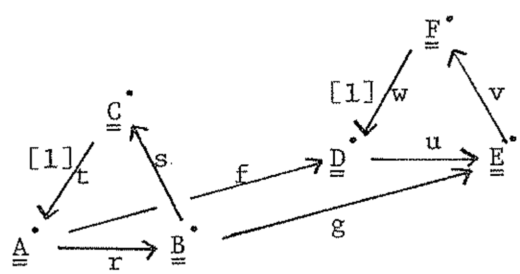
plex, we see that every triangle is isomorphic to a triangle of injective complexes.

Theorem 4.10: (i) $D^b(X)$ is a triangulated category and theorems 4.1-4.9 hold for $D^b(X)$ in place of $K^b(X)$;

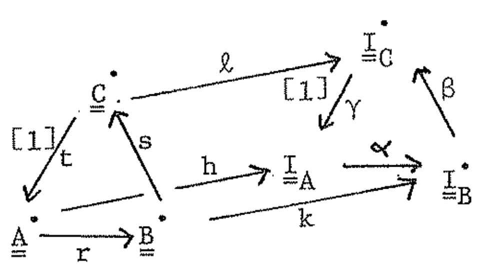
(ii) $K^b_f(X)$ and $D^b_f(X)$ are triangulated categories and the analogues of theorems 4.1-4.9 hold for these.

Proof: (i) It's clear that every map $\underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet$ is the bottom of some triangle, and that every diagram isomorphic to a triangle is a triangle. Each of the theorems 4.1-4.9 follows immediately from the corresponding theorem in $K^b(X)$ by mapping triangles in $D^b(X)$ into isomorphic triangles of injective complexes in $K^b(X)$. For example, the proof of theorem 4.1 is as follows.

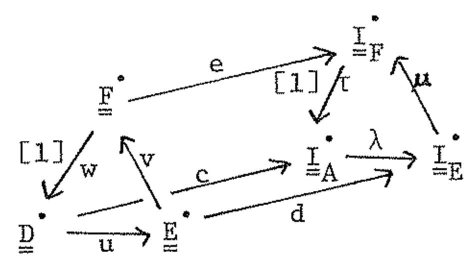
Let



be a diagram in $D^b(X)$ involving two triangles. Take isomorphisms



and



of the triangles into triangles of injective complexes. Then there are maps $\sigma : \underline{\underline{I}}_A^\bullet \to \underline{\underline{I}}_D^\bullet$, $\rho : \underline{\underline{I}}_B^\bullet \to \underline{\underline{I}}_E^\bullet$ such that $\rho\alpha = \lambda\sigma$, $\rho k = dg$, and $\sigma h = ef$. But since $D^b(X)$ maps between injective complexes are given by maps in $K^b(X)$, we can apply theorem 4.1 to get an $\eta : \underline{\underline{I}}_C^\bullet \to \underline{\underline{I}}_F^\bullet$, giving a map between the two injective triangles. Then a $D^b(X)$ map $\theta : \underline{\underline{C}}^\bullet \to \underline{\underline{F}}^\bullet$ making (f,g,θ) a map of triangles can be defined by $e^{-1}\eta\ell$.

(ii) In both $K^b(X)$ and $D^b(X)$, if $f : \underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet$ is a map between complexes of sheaves with finite dimensional stalks, then there is a triangle

$$\begin{tikzcd} & \underline{\underline{C}}^\bullet \arrow[dl, "{[1]}"'] & \\ \underline{\underline{A}}^\bullet \arrow[rr, "f"] & & \underline{\underline{B}}^\bullet \arrow[ul] \end{tikzcd}$$

where $\underline{\underline{C}}^\bullet$ also is a complex of sheaves with finite dimensional stalks, since injective resolutions with finite dimensional stalks can be taken, and the mapping cone of a map between such complexes clearly has finite dimensional stalks. Then (ii) follows from this and the fact that $K_f^b(X) \subseteq K^b(X)$ and $D_f^b(X) \subseteq D^b(X)$ can be considered as full subcategories.

It should be noted that except for theorem 4.8, each of theorems 4.1-4.9 is valid in any triangulated category.

In the same way as with $K^b(X)$, applying $\underline{\underline{H}}^0$ to a triangle in $D^b(X)$ gives us a sequence of sheaves, and it's clear that this sequence is exact.

<u>Theorem 4.11</u>: $0 \to \underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet \to 0$ is a short exact sequence of bounded complexes of sheaves on X and chain maps (<u>not</u> necessarily split), then there is a triangle in $D^b(X)$,

$$\begin{tikzcd} & \underline{\underline{C}}^\bullet \arrow[dl, "{[1]}"'] & \\ \underline{\underline{A}}^\bullet \arrow[rr] & & \underline{\underline{B}}^\bullet \arrow[ul] \end{tikzcd}$$

whose two degree 0 maps are represented by the given chain maps, and whose associated long exact sequence of sheaves is the same as the one given by the short exact sequence.

<u>Proof</u>: We can resolve the sequence into a short exact sequence of injective complexes and chain maps

$$\begin{tikzcd} 0 \arrow[r] & \underline{\underline{A}}^\bullet \arrow[r] \arrow[d] & \underline{\underline{B}}^\bullet \arrow[r] \arrow[d] & \underline{\underline{C}}^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \underline{\underline{I}}^\bullet \arrow[r] & \underline{\underline{J}}^\bullet \arrow[r] & \underline{\underline{K}}^\bullet \arrow[r] & 0 \end{tikzcd}$$

(note that the construction of theorem 1.3.4 is functorial with respect to chain maps, and not just up to homotopy). Since $\underline{\underline{I}}^\bullet$ is injective, this new sequence is split, and hence induces a triangle in $K^b(X)$ giving the same long exact sequence by theorem 4.8. Then the diagram $\underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet \to \underline{\underline{A}}^\bullet[1]$ in $D^b(X)$ whose maps are given by the maps of the triangle $\underline{\underline{I}}^\bullet \to \underline{\underline{J}}^\bullet \to \underline{\underline{K}}^\bullet \to \underline{\underline{I}}^\bullet[1]$ is then a triangle having the desired properties.

If $F : S(X) \to S(Y)$ is a functor, then we have an induced functor $F^{\bullet} : K^b(X) \to K^b(Y)$; for F covariant, $F^i(\underline{\underline{S}}^{\bullet}) = F(\underline{\underline{S}}^i)$ and for F contravariant, $F^i(\underline{\underline{S}}^{\bullet}) = F(\underline{\underline{S}}^{-i})$. We also have similar statements for $S(X)$ and $S(Y)$ replaced with $S_f(X)$, $S_f(Y)$, VS , or VS_f .

<u>Theorem 4.12</u>: (i) Let $F : S(X) \to S(Y)$ be a functor preserving direct sums (i.e., $F(\underline{\underline{S}} \oplus \underline{\underline{T}}) = F(\underline{\underline{S}}) \oplus F(\underline{\underline{T}})$), and let $\underline{\underline{A}}^{\bullet} \overset{u}{\to} \underline{\underline{B}}^{\bullet} \overset{v}{\to} \underline{\underline{C}}^{\bullet} \overset{w}{\to} \underline{\underline{A}}^{\bullet}[1]$ be a triangle in $K^b(X)$. If F is covariant, then $F^{\bullet}\underline{\underline{A}}^{\bullet} \xrightarrow{Fu} F^{\bullet}\underline{\underline{B}}^{\bullet} \xrightarrow{Fv} F^{\bullet}\underline{\underline{C}}^{\bullet} \xrightarrow{Fw} F^{\bullet}\underline{\underline{A}}^{\bullet}[1]$ is a triangle in $K^b(Y)$ and if F is contravariant, then $F^{\bullet}\underline{\underline{C}}^{\bullet} \xrightarrow{-Fv} F^{\bullet}\underline{\underline{B}}^{\bullet} \xrightarrow{Fu} F^{\bullet}\underline{\underline{A}}^{\bullet} \xrightarrow{Fw} F^{\bullet}\underline{\underline{C}}^{\bullet}[1]$ is a triangle in $K^b(Y)$.

(ii) If $F : S(X) \to S(Y)$ is a covariant functor preserving direct sums, and $\underline{\underline{A}}^{\bullet} \to \underline{\underline{B}}^{\bullet} \to \underline{\underline{C}}^{\bullet} \to \underline{\underline{A}}^{\bullet}[1]$ is a triangle in $D^b(X)$, then $R^{\bullet}F\underline{\underline{A}}^{\bullet} \to R^{\bullet}F\underline{\underline{B}}^{\bullet} \to R^{\bullet}F\underline{\underline{C}}^{\bullet} \to R^{\bullet}F\underline{\underline{A}}^{\bullet}[1]$ is a triangle in $D^b(Y)$.

(iii) Similar statements hold if $S(X)$ and $S(Y)$ are replaced with $S_f(X)$, $S_f(Y)$, VS , or VS_f .

<u>Proof</u>: (i) The result for F covariant follows from the fact that $M(Fu^{\bullet}) = F^{\bullet}M(u^{\bullet})$. For F contravariant, we see that $F^{\bullet}(M(u^{\bullet}))[1] = M(F^{\bullet}u^{\bullet})$, hence we have a triangle

$$\begin{tikzcd} & F^{\bullet}\underline{\underline{C}}^{\bullet}[1] \arrow[dl, "{[1]}Fv"'] & \\ F^{\bullet}\underline{\underline{B}}^{\bullet} \arrow[rr, "Fu"'] & & F^{\bullet}\underline{\underline{A}}^{\bullet} \arrow[ul, "Fw"'] \end{tikzcd}$$

The result follows then from theorem 4.5.

To show (ii), resolve the triangle into a triangle of injective complexes in $K^b(X)$ and apply (i).

(iii) is clear, especially after noting that $VS = S(P)$ where P is a complex consisting of a single point.

Lemma 4.13: Let $\alpha^{\bullet\bullet} : \underline{\underline{A}}^{\bullet\bullet} \to \underline{\underline{B}}^{\bullet\bullet}$ be a map of double complexes, and let $\tilde{M}^{\bullet\bullet}(\alpha^{\bullet\bullet})$ be the double complex formed by letting the slice $\tilde{M}^{\bullet j}(\alpha^{\bullet\bullet})$ be the mapping cone of $\alpha^{\bullet j}\ \underline{\underline{A}}^{\bullet j} \to \underline{\underline{B}}^{\bullet j}$. Then we have a diagram of double complexes

$$\begin{tikzcd} & M^{\bullet\bullet}(\alpha^{\bullet\bullet}) \arrow[dl, "{[1,0]}"', "s^{\bullet\bullet}"] & \\ \underline{\underline{A}}^{\bullet\bullet} \arrow[rr, "\alpha^{\bullet\bullet}"] & & \underline{\underline{B}}^{\bullet\bullet} \arrow[ul, "t^{\bullet\bullet}"'] \end{tikzcd}$$

where $s^{ij} : \tilde{M}^{ij}(\alpha^{\bullet\bullet}) \to \underline{\underline{A}}^{i+1,j}$, and this induces a triangle when associated single complexes are taken.

Proof: If $u^{\bullet} : \underline{\underline{S}}^{\bullet} \to \underline{\underline{T}}^{\bullet}$ is a chain map, we can form a double complex $\underline{\underline{R}}^{\bullet\bullet}$ by letting $\underline{\underline{R}}^{\bullet,-1}$ be $\underline{\underline{S}}^{\bullet}$, $\underline{\underline{R}}^{\bullet 0}$ be $\underline{\underline{T}}^{\bullet}$, and $\underline{\underline{R}}^{\bullet j}$ be the 0-complex for $j = -1$ or 0. Then $M(u^{\bullet})$ is the associated single complex of $\underline{\underline{R}}^{\bullet\bullet}$ for an appropriate sign convention. If $\underline{\underline{S}}^{\bullet\bullet}$ is the double complex with $\underline{\underline{S}}^{\bullet 0} = \underline{\underline{S}}^{\bullet}$ and $\underline{\underline{S}}^{ij} = 0$ for $j \neq 0$, and $\underline{\underline{T}}^{\bullet\bullet}$ is similarly defined, then we have a diagram of double complexes

$$\begin{tikzcd} & \underline{\underline{R}}^{\bullet\bullet} \arrow[dl, "{[0,1]}"'] & \\ \underline{\underline{S}}^{\bullet\bullet} \arrow[rr] & & \underline{\underline{T}}^{\bullet\bullet} \arrow[ul] \end{tikzcd}$$

, and this gives us the triangle

$$\begin{tikzcd} & M(u^{\bullet}) \arrow[dl, "{[1]}"'] & \\ \underline{\underline{S}}^{\bullet} \arrow[rr] & & \underline{\underline{T}}^{\bullet} \arrow[ul] \end{tikzcd}$$

when associated single complexes are taken.

In the same way, we can form a triple complex $\underline{\underline{R}}^{\bullet\bullet\bullet}$ from $\alpha^{\bullet\bullet} : \underline{\underline{A}}^{\bullet\bullet} \to \underline{\underline{B}}^{\bullet\bullet}$ with $\underline{\underline{R}}^{\bullet\bullet,-1} = \underline{\underline{A}}^{\bullet\bullet}$, $\underline{\underline{R}}^{\bullet\bullet,0} = \underline{\underline{B}}^{\bullet\bullet}$, and $\underline{\underline{R}}^{ijk} = 0$ for $k \neq -1$ or 0. Let $\underline{\underline{A}}^{\bullet\bullet\bullet}$ be the triple complex with $\underline{\underline{A}}^{\bullet\bullet 0} = \underline{\underline{A}}^{\bullet\bullet}$, $\underline{\underline{A}}^{ijk} = 0$ for $k \neq 0$, and let $\underline{\underline{B}}^{\bullet\bullet\bullet}$ be defined similarly. We then have a diagram of triple complexes,

$$\begin{tikzcd} & \underline{\underline{R}}^{\cdots} \arrow[dl, "{[0,0,1]}"'] & \\ \underline{\underline{A}}^{\cdots} \arrow[rr] & & \underline{\underline{B}}^{\cdots} \arrow[ul] \end{tikzcd}$$

If we reduce these complexes in the first two coordinates we get a diagram of double complexes of the same type, and by the comments at the beginning of the proof, reducing this further to a diagram of single complexes produces a triangle.

If we reduce the triple complexes in the first and third coordinate, however, we get the diagram of double complexes described in the statement of the lemma, so by theorems 1.2.1 and 1.2.2, when we reduce this to a diagram of single complexes, the result is isomorphic to the diagram acquired by the first pair of reductions. The lemma then follows.

<u>Theorem 4.14</u>: (i) If $\underline{\underline{A}}^{\bullet} \to \underline{\underline{B}}^{\bullet} \to \underline{\underline{C}}^{\bullet} \to \underline{\underline{A}}^{\bullet}[1]$ is a triangle in $K^b(X)$ or $D^b(X)$, then we have a triangle

$$\begin{tikzcd} & \underline{\underline{C}}^{\bullet} \otimes \underline{\underline{D}}^{\bullet} \arrow[dl, "{[1]}"'] & \\ \underline{\underline{A}}^{\bullet} \otimes \underline{\underline{D}}^{\bullet} \arrow[rr] & & \underline{\underline{B}}^{\bullet} \otimes \underline{\underline{D}}^{\bullet} \arrow[ul] \end{tikzcd}$$

for a bounded complex $\underline{\underline{D}}^{\bullet}$, where the maps are the naturally induced ones.

(ii) Given $\underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet \to \underline{\underline{A}}^\bullet[1]$ a triangle in $K^b(X)$ and $\underline{\underline{D}}^\bullet \in K^b(X)$, we have triangles

$$\begin{tikzcd}
 & \underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{D}}^\bullet, \underline{\underline{C}}^\bullet) \arrow[dl, "{[1]}"'] & \\
\underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{D}}^\bullet, \underline{\underline{A}}^\bullet) \arrow[rr] & & \underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{D}}^\bullet, \underline{\underline{B}}^\bullet) \arrow[ul]
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{A}}^\bullet, \underline{\underline{D}}^\bullet) \arrow[dl, "{[1]}"'] & \\
\underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{C}}^\bullet, \underline{\underline{D}}^\bullet) \arrow[rr, "-"] & & \underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{B}}^\bullet, \underline{\underline{D}}^\bullet) \arrow[ul]
\end{tikzcd}$$

, where the map labeled - is the negative of the induced one; all others are the naturally induced map.

(iii) Given $\underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet \to \underline{\underline{A}}^\bullet[1]$ a triangle in $D^b(X)$ and $\underline{\underline{D}}^\bullet \in D^b(X)$, we have the same triangles as in (ii), but with $\underline{\underline{\mathrm{Hom}}}^\bullet$ replaced with $R^\bullet\underline{\underline{\mathrm{Hom}}}$.

Proof: (i) For the triangle in $K^b(X)$, assume it is of the form $\underline{\underline{A}}^\bullet \xrightarrow{u} \underline{\underline{B}}^\bullet \to M(u^\bullet) \to \underline{\underline{A}}^\bullet[1]$. Then for $\underline{\underline{D}}^i$ we have a triangle $\underline{\underline{A}}^\bullet \otimes \underline{\underline{D}}^i \to \underline{\underline{B}}^\bullet \otimes \underline{\underline{D}}^i \to M(u^\bullet) \otimes \underline{\underline{D}}^i \to \underline{\underline{A}}^\bullet \otimes \underline{\underline{D}}^i[1]$ with $M(u^\bullet) \otimes \underline{\underline{D}}^i$ canonically isomorphic to $M(u^\bullet \otimes \mathrm{id}_{\underline{\underline{D}}^i})$, so lemma 4.13 applies. If the triangle is in $D^b(X)$, we may replace it with an isomorphic triangle in $K^b(X)$ and use the result for $K^b(X)$.

(ii) The diagram acquired by applying $\underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{D}}^\bullet,\cdot)$ is a triangle by the same reasoning as was used with the functor $\cdot \otimes \underline{\underline{D}}^\bullet$. If $\underline{\underline{\mathrm{Hom}}}^\bullet(\cdot,\underline{\underline{D}}^\bullet)$ is applied to the triangle, we get a diagram

$$\begin{tikzcd}
& \underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{C}}^\bullet,\underline{\underline{D}}^\bullet)[1] \arrow[dl, "{[1]}"'] & \\
\underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{B}}^\bullet,\underline{\underline{D}}^\bullet) \arrow[rr] & & \underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{A}}^\bullet,\underline{\underline{D}}^\bullet) \arrow[ul]
\end{tikzcd}$$

If we assume $\underline{\underline{C}}^\bullet = M(u^\bullet)$ for a chain map $u^\bullet : \underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet$, then as was seen in the proof of theorem 4.12, $\underline{\underline{\mathrm{Hom}}}^\bullet(M(u^\bullet),\underline{\underline{D}}^i)[1]$ is canonically isomorphic to $M(\underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{B}}^\bullet,\underline{\underline{D}}^i) \to \underline{\underline{\mathrm{Hom}}}^\bullet(\underline{\underline{A}}^\bullet,\underline{\underline{D}}^i))$ for each i, so lemma 4.12 applies again. Hence the above diagram is a triangle, so the result follows by theorem 4.5.

(iii) follows from (ii) by taking injective resolutions in $K^b(X)$.

<u>Theorem 4.15</u>: Given a cellular map $f : X \to Y$ and a triangle $\underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet \to \underline{\underline{C}}^\bullet \to \underline{\underline{A}}^\bullet[1]$ in $D^b(Y)$, then $f^!\underline{\underline{A}}^\bullet \to f^!\underline{\underline{B}}^\bullet \to f^!\underline{\underline{C}}^\bullet \to f^!\underline{\underline{A}}^\bullet[1]$ is a triangle in $D^b(X)$.

<u>Proof</u>: This follows the fact that both f^* and $R^\bullet\underline{\underline{\mathrm{Hom}}}(\cdot,\underline{\underline{D}}^\bullet)$ preserve triangles, since $f^! = Df^*D$.

<u>Theorem 4.16</u>: For $\underline{\underline{A}}^\bullet \in D^b(X)$, there is a triangle

$$\begin{tikzcd}[column sep=small] & \tau_{\le p}\underline{\underline{A}}^\bullet \arrow[dl, "{[1]}"'] & \\ \tau^{\ge p+1}\underline{\underline{A}}^\bullet \arrow[rr] & & \underline{\underline{A}}^\bullet \arrow[ul] \end{tikzcd}$$

where the two degree 0 maps are the natural ones.

<u>Proof</u>: There is a short exact sequence of complexes of sheaves and chain maps $0 \to \tau_{\le p}\underline{\underline{A}}^\bullet \to \underline{\underline{A}}^\bullet \to \tilde{\tau}^{\ge p+1}\underline{\underline{A}}^\bullet \to 0$, and this induces a triangle $\tau_{\le p}\underline{\underline{A}}^\bullet \to \underline{\underline{A}}^\bullet \to \tilde{\tau}^{\ge p+1}\underline{\underline{A}}^\bullet \to \tau_{\le p}\underline{\underline{A}}^\bullet[1]$ in $D^b(X)$ by theorem 4.11. Since $\tau^{\ge p+1}$ and $\tilde{\tau}^{\ge p+1}$ both satisfy the conditions of lemma 2.4.1, this triangle is isomorphic to a triangle $\tau_{\le p}\underline{\underline{A}}^\bullet \to \underline{\underline{A}}^\bullet \to \tau^{\ge p+1}\underline{\underline{A}}^\bullet \to \tau_{\le p}\underline{\underline{A}}^\bullet[1]$ with the degree 0 maps being the natural ones.

<u>Definition</u>: Let $Y \subseteq X$ be a subset of cells (not necessarily a subcomplex), such that if $\sigma \le \tau \le \gamma$ are in X and $\sigma,\gamma \in Y$, then $\tau \in Y$ (Y is called a <u>locally</u> <u>closed</u> subset). Define $r_Y : D^b(X) \to D^b(X)$ by letting $r_Y\underline{\underline{A}}^\bullet$ be the complex which in degree i is the sheaf having $\underline{\underline{A}}^i(\sigma)$ on σ for $\sigma \in Y$ and 0 for $\sigma \notin Y$. Define $r^Y : D^b_f(X) \to D^b_f(X)$ to be Dr_YD.

If $i : Y \hookrightarrow X$ is the inclusion of a subcomplex, then it's clear that $r_Y\underline{\underline{A}}^\bullet = i_!i^*\underline{\underline{A}}^\bullet \overset{qi}{=} Ri_!i^*\underline{\underline{A}}^\bullet$ (the last step since for i an inclusion, i is fibred and $i_!$ is exact), and hence
$r^Y\underline{\underline{A}}^\bullet = Dr_YD\underline{\underline{A}}^\bullet \overset{qi}{=} DRi_!DDi^*D\underline{\underline{A}}^\bullet \overset{qi}{=} Ri_*i^!\underline{\underline{A}}^\bullet$.

If $Y \subseteq X$ is locally closed and $R \subseteq Y$ is a closed subset of Y

(i.e., for $\sigma, \tau \in Y$, $\sigma \leq \tau \in R = \sigma \in R$), then there is a natural map $r_Y\underline{\underline{A}}^\bullet \to r_R\underline{\underline{A}}^\bullet$, and if $R \subseteq Y$ is open in Y (i.e., $Y - R$ is closed in Y), there is a natural map $r_R\underline{\underline{A}}^\bullet \to r_Y\underline{\underline{A}}^\bullet$. Hence we also have natural maps $r^R\underline{\underline{A}}^\bullet \to r^Y\underline{\underline{A}}^\bullet$ acquired by applying D to $r_Y D\underline{\underline{A}}^\bullet \to r_R D\underline{\underline{A}}^\bullet$ for R closed in Y, and $r^Y\underline{\underline{A}}^\bullet \to r^R\underline{\underline{A}}^\bullet$ acquired in a similar way for R open in Y.

Theorem 4.17: For $Y \subseteq X$ locally closed and $R \subseteq Y$ closed in Y, we have the following triangles, where the degree 0 maps are the natural ones:

$$\begin{tikzcd} & r_R\underline{\underline{A}}^\bullet \arrow[dl, "{[1]}"'] & \\ r_{Y-R}\underline{\underline{A}}^\bullet \arrow[rr] & & r_Y\underline{\underline{A}}^\bullet \arrow[ul] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & r^{Y-R}\underline{\underline{A}}^\bullet \arrow[dl, "{[1]}"'] & \\ r^R\underline{\underline{A}}^\bullet \arrow[rr] & & r^Y\underline{\underline{A}}^\bullet \arrow[ul] \end{tikzcd} .$$

Proof: The first triangle is obtained from the short exact sequence of chain maps

$$0 \to r_{Y-R}\underline{\underline{A}}^\bullet \to r_Y\underline{\underline{A}}^\bullet \to r_R\underline{\underline{A}}^\bullet \to 0 .$$

The second triangle is obtained by constructing the first triangle for the complex $D\underline{\underline{A}}^\bullet$, and then applying D. This actually gives a triangle with the map $r^R\underline{\underline{A}}^\bullet \to r^Y\underline{\underline{A}}^\bullet$ being the negative of the natural one, but by using the isomorphism $r^R\underline{\underline{A}}^\bullet \to r^R\underline{\underline{A}}^\bullet$ that multiplies by -1, an isomorphic triangle can be formed which uses the natural map

$r^R\underline{\underline{A}}^\bullet \to r^Y\underline{\underline{A}}^\bullet$.

Remark: For $\underline{\underline{I}}^\bullet$ an injective complex, $r^Y\underline{\underline{I}}^\bullet$ is simply the complex $\underline{\underline{I}}^\bullet$, but with the elementary injectives $[\sigma]^V$ for $\sigma \notin Y$ removed. For Y a subcomplex of X , this follows from theorem 3.3.5; for Y not a subcomplex, the proof of this is similar to the proof of theorem 3.3.5.

Under this interpretation, for $Y \subseteq X$ locally closed, $R \subseteq Y$ closed in Y , and $\underline{\underline{I}}^\bullet$ an injective complex, the map $r^R\underline{\underline{I}}^\bullet \to r^Y\underline{\underline{I}}^\bullet$ is just the obvious (chain map) inclusion, and $r^Y\underline{\underline{I}}^\bullet \to r^{Y-R}\underline{\underline{I}}^\bullet$ the obvious surjection. The second triangle in theorem 4.17 can be obtained, then, from the short exact sequence of chain maps,

$$0 \to r^R\underline{\underline{I}}^\bullet \to r^Y\underline{\underline{I}}^\bullet \to r^{Y-R}\underline{\underline{I}}^\bullet \to 0 \ .$$

Theorem 4.18: Given a triangle
$$\begin{tikzcd} & \underline{\underline{C}}^\bullet \arrow[dl, "{[1]}"'] & \\ \underline{\underline{A}}^\bullet \arrow[rr] & & \underline{\underline{B}}^\bullet \arrow[ul] \end{tikzcd}$$
in $D^b(X)$ and $D \in D^b(X)$, we have long exact sequences of the form

$$\cdots \to F^{i-1}(\underline{\underline{C}}^\bullet) \to F^i(\underline{\underline{A}}^\bullet) \to F^i(\underline{\underline{B}}^\bullet) \to F^i(\underline{\underline{C}}^\bullet) \to F^{i+1}(\underline{\underline{A}}^\bullet) \to \cdots$$

for $F^i = \underline{\underline{H}}^i, \mathbb{H}^i, \mathbb{H}^i_c$, $\underline{\underline{\underline{Ext}}}^i(\underline{\underline{D}}^\bullet,\cdot)$, and $\mathrm{Ext}^i(\underline{\underline{D}}^\bullet,\cdot)$, and of the form

$$\cdots \to F^{i-1}(\underline{\underline{A}}^\bullet) \to F^i(\underline{\underline{C}}^\bullet) \to F^i(\underline{\underline{B}}^\bullet) \to F^i(\underline{\underline{A}}^\bullet) \to F^{i+1}(\underline{\underline{C}}^\bullet) \to \cdots$$

for $F^i = \underline{\underline{\underline{Ext}}}^i(\cdot,\underline{\underline{D}}^\bullet)$ or $Ext^i(\cdot,\underline{\underline{D}}^\bullet)$.

$$\begin{tikzcd} & \underline{\underline{C}}^\bullet \arrow[dl, "{[1]}"'] & \\ \underline{\underline{A}}^\bullet \arrow[rr] & & \underline{\underline{B}}^\bullet \arrow[ul] \end{tikzcd}$$

For $\underline{\underline{A}}^\bullet \to \underline{\underline{B}}^\bullet$ a triangle in $K^b(X)$ and $\underline{\underline{D}}^\bullet \in K^b(X)$, there is a long exact sequence of the first form for $F^i = H^i\underline{\underline{\underline{Hom}}}^\bullet(\underline{\underline{D}}^\bullet,\cdot)$ or $H^i Hom^\bullet(\underline{\underline{D}}^\bullet,\cdot)$ and of the second form for $F^i = H^i\underline{\underline{\underline{Hom}}}^\bullet(\cdot,\underline{\underline{D}}^\bullet)$ or $H^i Hom^\bullet(\cdot,\underline{\underline{D}}^\bullet)$.

<u>Proof</u>: The case $R^i = \underline{\underline{H}}^i$ has already been shown. Interpreting X as a point space, we see that triangles in VS give long exact sequences when $H^\bullet$ is applied to them. Then the cases $R^i = \mathbb{H}^i, \mathbb{H}^i_c$, $\underline{\underline{\underline{Ext}}}^i(\underline{\underline{D}}^\bullet,\cdot)$, $Ext^i(\underline{\underline{D}}^\bullet,\cdot)$, $\underline{\underline{\underline{Ext}}}^i(\cdot,\underline{\underline{D}}^\bullet)$ and $Ext^i(\cdot,\underline{\underline{D}}^\bullet)$ follow from the first case using the triangles acquired by applying the functors $R^\bullet\Gamma$, $R^\bullet\Gamma_c$, $R^\bullet\underline{\underline{\underline{Hom}}}(\underline{\underline{D}}^\bullet,\cdot)$, $R^\bullet Hom(\underline{\underline{D}}^\bullet,\cdot)$, $R^\bullet\underline{\underline{\underline{Hom}}}(\cdot,\underline{\underline{D}}^\bullet)$ and $R^\bullet Hom(\cdot,\underline{\underline{D}}^\bullet)$ respectively, to the original triangle.

The cases for $K^b(X)$ are similar.

<u>Remark</u>: The sequences involving $Ext^i(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)$ and $H^i Hom(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet)$ are quite useful, since these represent $Hom_{D^b(X)}(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet[i])$ and $Hom_{K^b(X)}(\underline{\underline{A}}^\bullet,\underline{\underline{B}}^\bullet[i])$, respectively.

CHAPTER FIVE

THE EQUIVALENCE OF THE CELLULAR AND TOPOLOGICAL CONSTRUCTIBLE DERIVED CATEGORIES

§5.1 Constructibility in $D^b(X)$

Let X be a cell complex, and let $\mathcal{S}$ be a sequence of closed subcomplexes $X^0 \subseteq X^1 \subseteq \ldots \subseteq X^n = X$. Then each subset $X^i - X^{i-1} \subseteq X$, called a stratum of $\mathcal{S}$, is locally closed. A sheaf $\underline{\underline{A}} \in S_f(X)$ is called constructible with respect to $\mathcal{S}$ if given $\sigma,\tau \in X^i - X^{i-1}$ with $\sigma \leq \tau$, the corestriction map $p^{\underline{\underline{A}}}_{\sigma,\tau}$ is an isomorphism. We define the constructible derived category $D^b_{\mathcal{S}}(X) \subseteq D^b_f(X)$ to be the full subcategory consisting of complexes $\underline{\underline{A}}^\bullet$ for which $H^i\underline{\underline{A}}^\bullet$ is constructible for each i. Note that by theorem 3.1.3, $D^b_{\mathcal{S}}(X)$ is equivalent to the full subcategory of $D^b_{fc}(X)$ of complexes $\underline{\underline{A}}^\bullet$ whose stalk cohomology sheaves are constructible.

Theorem 5.1.1: The triangles of $D^b_f(X)$ give $D^b_{\mathcal{S}}(X)$ the structure of a triangulated category, and theorems 4.1-4.9 and 4.11 are valid in $D^b_{\mathcal{S}}(X)$.

Proof: Each of the above theorems, and hence the fact that $D^b_{\mathcal{S}}(X)$ is a triangulated category, follows from the fact that for any triangle in $D^b_f(X)$, if two of the objects are in $D^b_{\mathcal{S}}(X)$, then the third is, as well. To show this, take a triangle with two elements in $D^b_{\mathcal{S}}(X)$, and apply $\underline{\underline{H}}^\bullet$, getting a long exact sequence. For $\sigma \leq \tau$ in a stratum, the corestriction maps $p_{\sigma,\tau}$ of the stalk cohomology sheaves give a map between two long exact sequences of vector spaces. It follows from the 5-lemma that all the corestriction maps $p_{\sigma,\tau}$ are isomorphisms, so the result follows.

<u>Theorem 5.1.2</u>: (i) The functors $\otimes$, $\tau_{\leq p}$, $\tau^{\geq p}$, and r_Y for $Y \subseteq X$ a union of strata map objects of $D^b_{\mathscr{S}}(X)$ to objects of $D^b_{\mathscr{S}}(X)$.

(ii) For $i : Y \hookrightarrow X$ an inclusion, Y a union of strata, $Ri_!$ maps $D^b_{\mathscr{S}'}(Y)$ to $D^b_{\mathscr{S}}(X)$, the strata of $\mathscr{S}'$ being those of $\mathscr{S}$.

(iii) For $f : X \to Y$, X with strata given by $\mathscr{S}$ and Y with strata given by $\mathscr{S}'$, f mapping each stratum of X into a stratum of Y , then f^* maps $D^b_{\mathscr{S}'}(Y)$ into $D^b_{\mathscr{S}}(X)$ (for example, f can be an inclusion of a closed union of strata).

<u>Proof</u>: The statement for $\tau_{\leq p}$, $\tau^{\geq p}$, r_Y , and f^* is clear. The statement for $Ri_!$ follows since $i_!$ is exact, so $Ri_! = i_! =$ extension by zero.

To show that $\underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet \in D^b_{\mathscr{S}}(X)$ if $\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet \in D^b_{\mathscr{S}}(X)$, note that the statement is clear for $\underline{\underline{A}}$ and $\underline{\underline{B}}$ single constructible sheaves. The statement is then shown for $\underline{\underline{A}} \otimes \underline{\underline{B}}^\bullet$ with $\underline{\underline{A}}$ a constructible sheaf and

$$\begin{tikzcd} & \underline{\underline{A}}\otimes\tau_{\leq p}\underline{\underline{B}}^\bullet \arrow[dl, "{[1]}"'] & \\ \underline{\underline{A}}\otimes\tau^{\geq p+1}\underline{\underline{B}}^\bullet \arrow[rr] & & \underline{\underline{A}}\otimes\underline{\underline{B}}^\bullet \arrow[ul] \end{tikzcd}$$

$\underline{\underline{B}}^\bullet \in D^b_{\mathscr{S}}(X)$ by using the triangle above in $D^b_f(X)$ and induction on the maximum value of $u - t$ for which $\underline{\underline{H}}^u\underline{\underline{B}}^\bullet$ and $\underline{\underline{H}}^t\underline{\underline{B}}^\bullet$ are both non-zero (by induction, $\underline{\underline{A}} \otimes \tau_{\leq p}\underline{\underline{B}}^\bullet$ and $\underline{\underline{A}} \otimes \tau^{\geq p+1}\underline{\underline{B}}^\bullet$ are in

$D^b_{\mathscr{S}}(X)$ for an appropriate p, hence $\underline{\underline{A}} \otimes \underline{\underline{B}}^\bullet$ is). The initial case in the induction is the case $\underline{\underline{A}} \otimes \underline{\underline{B}}$ for $\underline{\underline{A}},\underline{\underline{B}}$ constructible sheaves. To show the statement for a general tensor product $\underline{\underline{A}}^\bullet \otimes \underline{\underline{B}}^\bullet$, do a similar induction on the maximum value of $u - t$ for which $\underline{\underline{H}}^u\underline{\underline{A}}^\bullet$ and $\underline{\underline{H}}^t\underline{\underline{A}}^\bullet$ are both non-zero.

More conditions on $\mathscr{S}$ will be needed to insure that $R^\bullet\underline{\underline{\mathrm{Hom}}}$ takes elements of $D^b_{\mathscr{S}}(X)$ to elements of $D^b_{\mathscr{S}}(X)$, and that the dualizing complex is constructible. It will be seen in §5.2 that if $|X^0| \subseteq |X^1| \subseteq \dots \subseteq |X^n| = |X|$ is a stratification of a pseudomanifold then these statements are true.

If $Y \subseteq X$ is locally closed, then in the same way as we do with Y a cell complex, we can define the notion of a sheaf on Y. We also can define the cohomology with compact support $H^i_c(Y,\underline{\underline{S}})$ of such a sheaf in the same way as with a cell complex, i.e., to be the cohomology of the chain complex

$$\cdots \to \bigoplus_{\gamma^i \in Y} \underline{\underline{S}}(\gamma^i) \to \bigoplus_{\gamma^{i+1} \in Y} \underline{\underline{S}}(\gamma^{i+1}) \to \cdots .$$

Note that $H^i_c(Y,\underline{\underline{S}}) = H^i_c(X,\tilde{\underline{\underline{S}}})$, where $\tilde{\underline{\underline{S}}}$ is the extension by zero of $\underline{\underline{S}}$ to a sheaf on X.

A sheaf on a locally closed set $Y \subseteq X$ is called a <u>local system</u> if all of the corestriction maps are isomorphisms.

Theorem 5.1.3: Let $\mathscr{S}$ be a sequence of closed subcomplexes $X^0 \subseteq \ldots \subseteq X^n = X$ on X. Suppose the following condition is satisfied:

Given $\sigma \leq \tau$ in a stratum $X^i - X^{i-1}$ and a local system $\underline{\underline{S}}$ on a stratum $X^j - X^{j-1}$, then letting $C_\sigma = (X^j - X^{j-1}) \cap st(\sigma)$, C_τ defined similarly, we have that the natural map $H^i_c(C_\tau, \underline{\underline{S}}|_{C_\tau}) \to H^i_c(C_\sigma, \underline{\underline{S}}|_{C_\sigma})$ is an isomorphism for each i.

Then $\underline{\underline{\mathbb{D}}}^\bullet_X \in D^b_{\mathscr{S}}(X)$ and $R^\bullet\underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet) \in D^b_{\mathscr{S}}(X)$ for any $\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet \in D^b_{\mathscr{S}}(X)$.

Proof: Let $\underline{\underline{S}}$ be a local system on $X^j - X^{j-1}$, and $\tilde{\underline{\underline{S}}}$ be the extension by zero to X. It can be directly verified that $D\tilde{\underline{\underline{S}}}$ being in $D^b_{\mathscr{S}}(X)$ is equivalent to the statement that for $\sigma \leq \tau$ in a stratum $X^i - X^{i-1}$, and C_τ and C_σ as above, the adjoint map

$$C^\bullet_c(C_\sigma, \underline{\underline{S}}|_{C_\sigma})^* \to C^\bullet_c(C_\tau, \underline{\underline{S}}|_{C_\tau})^*$$

is a quasi-isomorphism where $C^\bullet_c(C_\sigma, \underline{\underline{S}}|_{C_\sigma})$ is the chain complex that computes $H^\bullet_c(C_\sigma, \underline{\underline{S}}|_{C_\sigma})$ (and similarly for $C^\bullet_c(C_\tau, \underline{\underline{S}}|_{C_\tau})$). Then by the exactness of $V \longmapsto V^*$, $D\tilde{\underline{\underline{S}}} \in D^b_{\mathscr{S}}(X)$.

Next we show that $D\underline{\underline{A}}$ is in $D^b_{\mathscr{S}}(X)$ for $\underline{\underline{A}}$ constructible by induction on the number of strata on which $\underline{\underline{A}}$ is non-zero. We've shown the result for one non-zero stratum. Suppose there are k non-zero strata, the highest being $X^r - X^{r-1}$. Then $D\underline{\underline{A}} \in D^b_{\mathscr{S}}(X)$ by induction and the triangle

$$\begin{tikzcd} & Dr_{X^r-X^{r-1}}\underline{\underline{A}} \arrow[dl, "{[1]}"'] & \\ Dr_{X^{r-1}}\underline{\underline{A}} \arrow[rr] & & Dr_{X^r}\underline{\underline{A}}=D\underline{\underline{A}} \arrow[ul] \end{tikzcd}$$

To show that $D\underline{\underline{A}}^\bullet \in D^b_{\mathscr{S}}(X)$ for $\underline{\underline{A}}^\bullet \in D^b_{\mathscr{S}}(X)$, use the triangle

$$\begin{tikzcd} & D\tau^{\geq p+1}\underline{\underline{A}}^\bullet \arrow[dl, "{[1]}"'] & \\ D\tau_{\leq p}\underline{\underline{A}}^\bullet \arrow[rr] & & D\underline{\underline{A}}^\bullet \arrow[ul] \end{tikzcd}$$

and an induction similar to the type used in the proof of theorem 5.1.2 with tensor products.

The theorem now follows from the fact that $R^\bullet\underline{\underline{Hom}}(\underline{\underline{A}}^\bullet, \underline{\underline{B}}^\bullet)$ $\underline{\underline{qi}}$ $D(\underline{\underline{A}}^\bullet \otimes D\underline{\underline{B}}^\bullet)$ and that $\underline{\underline{\mathbb{D}}}_X^\bullet = D\underline{\underline{\mathbb{Q}}}_X$.

<u>Corollary 5.1.4</u>: Suppose the condition in theorem 5.1.3 is satisfied. Then

(i) $r^Y \underline{\underline{A}}^\bullet \in D^b(X)$ for $\underline{\underline{A}}^\bullet \in D^b(X)$

(ii) For $i : Y \quad X$ an inclusion, Y a union of strata, then Ri_* maps $D^b_{\mathscr{S}'}(Y)$ to $D^b_{\mathscr{S}}(X)$, the strata of $\mathscr{S}'$ being those of $\mathscr{S}$.

(iii) For $f : X \to Y$ as in theorem 5.1.2 (iii), $f^!$ maps $D^b_{\mathscr{S}'}(Y)$ into $D^b_{\mathscr{S}}(X)$.

<u>Proof:</u> These follow since $r^Y = Dr_Y D, R^\bullet f_* = DR^\bullet f_! D$, and $f^! = Df^* D$.

§5.2 <u>The Equivalence of $D^b_{\mathscr{S}}(X)$ and $D^b_{\mathscr{S}}(|X|)$.</u>

In this section we show how the theory developed in this paper corresponds to the standard theory of the derived category of the underlying space $|X|$. Some standard knowledge of sheaves and derived categories will be assumed; see, for instance, [GM], [H], or [Iv].

Let X be a cell complex and let $S(|X|)$ be the category of ordinary sheaves on $|X|$ (which will be referred to as topological sheaves). Define $S_{\mathscr{C}}(|X|) \subseteq S(|X|)$ to be sheaves that are locally constant on the cells of X. $D^b(|X|)$ will denote the bounded derived category of topological sheaves on $|X|$ (so objects are bounded chain complexes of elements of $S(|X|)$) and $D^b_{\mathscr{C}}(|X|) \subseteq D^b(|X|)$ is the full subcategory consisting of objects whose stalk cohomology sheaves are in $S\,(|X|)$ and have finite-dimensional stalks. $K^b_{\mathscr{C}}(|X|)$ and $K^b(|X|)$ are defined similarly.

Define a functor $\alpha : S(X) \to S(|X|)$ by letting $\alpha\underline{\underline{A}}$ be the sheafification of the presheaf which associates to $U \subseteq |X|$ the vector space $\Gamma(Y,\underline{\underline{A}})$ where Y is the union of all cells that have non-empty intersection with U. We can also define a functor $\underline{\underline{\Gamma}} : S(|X|) \to S(X)$ by $\underline{\underline{\Gamma A}}(\sigma) = \Gamma(|st(\sigma)|,\underline{\underline{A}})$, corestriction maps being restriction of sections. For $\underline{\underline{A}} \in S(X)$, there is a map $i_{\underline{\underline{A}}} : \underline{\underline{\Gamma}}\alpha\underline{\underline{A}} \to \underline{\underline{A}}$ which on σ is the map $\Gamma(st(\sigma),\underline{\underline{A}}) \to \underline{\underline{A}}(\sigma)$ given by evaluation, and for $\underline{\underline{B}} \in S(|X|)$ there is a map $V_{\underline{\underline{B}}} : \alpha\underline{\underline{\Gamma B}} \to \underline{\underline{B}}$ which on a stalk at $x \in \sigma \in X$ is the map $\Gamma(|st(\sigma)|,\underline{\underline{B}}) \to \underline{\underline{B}}_x$ given again by evaluation. These both commute with morphisms in $S(X)$ and $S(|X|)$, i.e., are natural transformations with the identity. $i_{\underline{\underline{A}}}$ is clearly an isomorphism for any $\underline{\underline{A}} \in S(X)$, and for $\underline{\underline{B}} \in S_{\mathscr{C}}(|X|)$, an element of $\underline{\underline{B}}_x$ for $x \in \sigma$ can be extended to $|st(\sigma)|$ by extending it (uniquely) a cell at a time, each cell having dimension greater or equal to the previous one, hence $V_{\underline{\underline{B}}}$ is also an isomorphism. Then we have that $S(X)$ and $S_{\mathscr{C}}(|X|)$ are equivalent where α and $\underline{\underline{\Gamma}}$ give equivalences in each direction.

α and $\underline{\underline{\Gamma}}$ induce functors $\alpha^{\bullet} : D^b(X) \to D^b(|X|)$ (since α is exact) and $R^{\bullet}\underline{\underline{\Gamma}} : D^b(|X|) \to D^b(X)$, and these restrict to functors $\alpha^{\bullet} : D^b_{fc}(X) \to D^b_{\mathscr{C}}(|X|)$ and $R^{\bullet}\underline{\underline{\Gamma}} : D^b_{\mathscr{C}}(|X|) \to D^b_{fc}(X)$. It will be shown that $\alpha^{\bullet}$ and $R^{\bullet}\Gamma$ are both equivalences of categories.

<u>Lemma 5.2.1</u>: Let $\sigma \in X$ and $\underline{\underline{A}}$ be a topological sheaf on $|st(\sigma)|$ which is locally constant on each cell. Then the Čech cohomology $\check{H}^i(|st(\sigma)|,\underline{\underline{A}})$ is 0 for $i > 0$.

<u>Proof</u>: Let $\sigma \leq \tau$, V be a vector space, and $j : |\tau| \hookrightarrow |\bar{\tau}|$,

$i : |\overline{\tau}| - |\tau| \hookrightarrow |\overline{\tau}|$ be inclusions (closures taken in $|st(\sigma)|$). We have a short exact sequence

$$0 \to j_! \underline{\underline{V}}_{|\tau|} \to \underline{\underline{V}}_{|\overline{\tau}|} \to i_* \underline{\underline{V}}_{|\overline{\tau}|-|\tau|} \to 0$$

(where $\underline{\underline{V}}_R$ denotes the constant sheaf with stalks V on the space R) which gives us a long exact sequence

$$\cdots \to \check{H}^{n-1}(|\overline{\tau}|, \underline{\underline{V}}_{|\overline{\tau}|}) \to \check{H}^{n-1}(|\overline{\tau}|, i_* \underline{\underline{V}}_{|\overline{\tau}|-|\tau|})$$

$$\to \check{H}^n(|\overline{\tau}|, j_! \underline{\underline{V}}_{|\tau|}) \to \check{H}^n(|\overline{\tau}|, \underline{\underline{V}}_{|\overline{\tau}|}) \to \cdots \quad .$$

Then $\check{H}^n(|\overline{\tau}|, j_! \underline{\underline{V}}_{|\tau|}) = 0$ for $n > 1$ since $|\overline{\tau}|$ and $|\overline{\tau}| - |\tau|$ are closed in $|\overline{\tau}|$ and topologically trivial (we're in $|st(\sigma)|$, so these contract to $|\sigma|$), which forces the 2^{nd} and 4^{th} groups to be zero. $H^1(|\overline{\tau}|, j_! \underline{\underline{V}}_{|\tau|}) = 0$ follows from the fact that the first map is an isomorphism for $i = 1$. If $k : |\tau| \hookrightarrow |st(\sigma)|$, $\ell : |\overline{\tau}| \hookrightarrow |st(\sigma)|$, then $\check{H}^n(|st(\sigma)|, k_! \underline{\underline{V}}_{|\tau|}) = \check{H}^n(|st(\sigma)|, \ell_* j_! \underline{\underline{V}}_{|\tau|}) = \check{H}^n(|st(\sigma)|, j_! \underline{\underline{V}}_{|\tau|}) = 0$ for $n > 0$. This proves the lemma for a sheaf constant on a single cell $|\tau|$ and 0 on other cells, and this implies the lemma for $\underline{\underline{A}}$ locally constant on $|\tau|$ and 0 elsewhere.

We now proceed by induction. Suppose the lemma is true for $\underline{\underline{A}}$ being non-zero on at most $m - 1$ cells, and let $\underline{\underline{A}}$ be non-zero on exactly m cells. Let τ be a cell of maximum dimension on which $\underline{\underline{A}}$ is non-zero. The lemma then follows from the long exact sequence associated to $0 \to k_! k^* \underline{\underline{A}} \xrightarrow{f} \underline{\underline{A}} \to \underline{\underline{\mathrm{Cok}}}(f) \to 0$ where $k : |\tau| \to |st(\sigma)|$, since $k_! k^* \underline{\underline{A}}$ and $\underline{\underline{\mathrm{Cok}}}(f)$ are non-zero on less than m cells.

Remark: Suppose X is a compact simplicial complex. Then lemma 5.2.1 shows that $\{|st(v)| \,|\, v$ is a vertex of $X\}$ is a Leray covering of $|X|$ since the set of intersections of these is $\{|st(\sigma)| \,|\, \sigma \in X\}$ and hence the Čech cohomology of a sheaf on $|X|$ can be computed using this covering, without taking further direct limits. It is easily verified that the Čech complex for a sheaf $\alpha\underline{\underline{A}}$ ($\underline{\underline{A}} \in S(X)$) using this covering is exactly the complex $C^{\bullet}(X,\underline{\underline{A}})$. Hence for compact simplicial complexes, cellular sheaf cohomology agrees with Čech cohomology. The result also holds for an arbitrary compact cell complex by taking a barycentric subdivision, and hence for any cell complex since it holds for the subcomplex $X' = \{\sigma \in X \,|\, |\bar{\sigma}|$ is compact$\}$ and X' is a deformation retract of X.

Lemma 5.2.2: For $\underline{\underline{I}}^{\bullet} \in K^b_{\mathscr{C}}(|X|)$ a complex of injective topological sheaves, let $V_{\underline{\underline{I}}^{\bullet}} : \alpha^{\bullet}\underline{\underline{\Gamma}}^{\bullet}\underline{\underline{I}}^{\bullet} \to \underline{\underline{I}}^{\bullet}$ be the map which is $V_{\underline{\underline{I}}^i}$ in degree i. Then $V_{\underline{\underline{I}}^{\bullet}}$ is a quasi-isomorphism.

Proof: We must show that for $x \in |X|$, $(V_{\underline{\underline{I}}^{\bullet}})_x : (\alpha^{\bullet}\underline{\underline{\Gamma}}^{\bullet}\underline{\underline{I}}^{\bullet})_x \to \underline{\underline{I}}^{\bullet}_x$ is a quasi-isomorphism, i.e., the evaluation map

$\Gamma(|st(\sigma)|, \underline{\underline{I}}^{\bullet}) \to \underline{\underline{I}}_x^{\bullet}$ for $x \in |\sigma|$ is a quasi-isomorphism. Since an injective restricted to $|st(\sigma)|$ is injective, we can assume that $\underline{\underline{I}}^{\bullet}$ is defined on $|st(\sigma)|$.

Let $\partial^i : \underline{\underline{I}}^i \to \underline{\underline{I}}^{i+1}$ be the boundary map. From the short exact sequence $0 \to \underline{\underline{Im}}\,\partial^{i-1} \to \underline{\underline{Ker}}\,\partial^i \to \underline{\underline{H}}^i\underline{\underline{I}}^{\bullet} \to 0$ we have the long exact sequence

$$\cdots \to H^j\underline{\underline{H}}^i\underline{\underline{I}}^{\bullet} \to H^{j+1}\underline{\underline{Im}}\,\partial^{i-1} \to H^{j+1}\underline{\underline{ker}}\,\partial^i \to H^{j+1}\underline{\underline{H}}^i\underline{\underline{I}}^{\bullet} \to \cdots ,$$

and hence $H^{j+1}\underline{\underline{Im}}\,\partial^{i-1} \to H^{j+1}\underline{\underline{ker}}\,\partial^i$ is an isomorphism for $j > 0$ since $\underline{\underline{H}}^i\underline{\underline{I}}^{\bullet}$ is locally constant on cells, so $H^j\underline{\underline{H}}^i\underline{\underline{I}}^{\bullet} = 0$ for $j > 0$ by lemma 5.2.1. From the short exact sequence $0 \to \underline{\underline{ker}}\,\partial^i \to \underline{\underline{I}}^i \to \underline{\underline{Im}}\,\partial^i \to 0$, we have the long exact sequence

$$\cdots \to H^j\underline{\underline{I}}^i \to H^j\underline{\underline{Im}}\,\partial^i \to H^{j+1}\underline{\underline{ker}}\,\partial^i \to H^{j+1}\underline{\underline{I}}^i \to \cdots ,$$

so $H^j\underline{\underline{Im}}\,\partial^i \to H^{j+1}\underline{\underline{ker}}\,\partial^i$ is an isomorphism for $j > 0$ since $\underline{\underline{I}}^i$ is injective. Then $H^{j+1}\underline{\underline{Im}}\,\partial^{i-1} \cong H^j\underline{\underline{Im}}\,\partial^i$, $j > 0$, so $H^1\underline{\underline{Im}}\,\partial^i \cong H^{k+1}\underline{\underline{Im}}\,\partial^{i-k} = 0$ for $k >> 0$. We also have that $H^1\underline{\underline{ker}}\,\partial^i = 0$ from $H^1\underline{\underline{Im}}\,\partial^{i-1} = 0$ and the first short exact sequence. We then have a short exact sequence of vector spaces $0 \to H^0\underline{\underline{\mathrm{Im}}}\partial^{i-1} \to H^0\underline{\underline{\mathrm{ker}}}\,\partial^i \to H^0\underline{\underline{H}}^i\underline{\underline{I}}^{\bullet} \to 0$. Since $\underline{\underline{H}}^i\underline{\underline{I}}^{\bullet}$ is locally constant on cells, we have an isomorphism $\Gamma(|st(\sigma)|,\underline{\underline{H}}^i\underline{\underline{I}}^{\bullet}) = H^0\underline{\underline{H}}^i\underline{\underline{I}}^{\bullet} \to \underline{\underline{H}}^i_x I^{\bullet}$ given by evaluation of the section at x , so we have a short exact sequence $0 \to H^0\underline{\underline{\mathrm{Im}}}\partial^{i-1} \to H^0\underline{\underline{\mathrm{ker}}}\partial^i \xrightarrow{\text{eval}} \underline{\underline{H}}^i_x\underline{\underline{I}}^{\bullet} \to 0$. Since $H^1\underline{\underline{\mathrm{ker}}}\partial^i = 0$, we also have short exact sequences of chain complexes

$$0 \to H^0\underline{\underline{\mathrm{ker}}}\partial^{\bullet} \to H^0\underline{\underline{I}}^{\bullet} \to H^0\underline{\underline{\mathrm{Im}}}\partial^{\bullet} \to 0$$

(the boundary maps in the first and third chain complex being 0), which gives an exact sequence

$$\cdots \to H^{i-1}(H^0\underline{\underline{I}}^{\bullet}) \to H^0\underline{\underline{\mathrm{Im}}}\,\partial^{i-1} \to H^0\underline{\underline{\mathrm{ker}}}\partial^i \to H^i(H^0\underline{\underline{I}}^{\bullet}) \to H^0\underline{\underline{\mathrm{Im}}}\partial^i \to \cdots \quad .$$

But, if $\gamma \in H^0\underline{\underline{I}}^k$ and $\gamma \longmapsto 0$ in $H^0\underline{\underline{I}}^{k+1}$, then $\gamma \in H^0\underline{\underline{\mathrm{ker}}}\partial^k$, so $H^k(H^0\underline{\underline{I}}^{\bullet}) \to H^0\underline{\underline{\mathrm{Im}}}\partial^k$ is the 0-map $\forall\, k$. Hence we have a short exact sequence

$$0 \to H^0\underline{\underline{\mathrm{Im}}}\,\partial^{i-1} \xrightarrow{\lambda} H^0\underline{\underline{\mathrm{ker}}}\partial^i \xrightarrow{\beta} H^i(H^0\underline{\underline{I}}^{\bullet}) \to 0 \ ,$$

where λ is induced by the inclusion map $\underline{\underline{\mathrm{Im}}}\partial^{i-1} \hookrightarrow \underline{\underline{\mathrm{ker}}}\,\partial^i$ and β is given by $\beta(\gamma) = [\gamma]$, interpreting γ as an element of $H^0\underline{\underline{I}}^{\bullet}$. Then the composition of β with the map $H^i(H^0\underline{\underline{I}}^\bullet) \to H^i\underline{\underline{I}}^\bullet_x$ induced by the evaluation map on sections is the map $\gamma \longmapsto [\gamma] \longmapsto [\gamma(x)]$, so we get a map between the two short exact sequences,

$$\begin{tikzcd}
0 \arrow[r] & H^0\underline{\underline{Im}}\,\partial^{i-1} \arrow[r] \arrow[d, "id"] & H^0\underline{\underline{ker}}\,\partial^i \arrow[r, "\beta"] \arrow[d, "id"] & H^i(H^0\underline{\underline{I}}^\bullet) \arrow[r] \arrow[d, "eval_*"] & 0 \\
0 \arrow[r] & H^0\underline{\underline{Im}}\,\partial^{i-1} \arrow[r] & H^0\underline{\underline{ker}}\,\partial^i \arrow[r, "\gamma \to [\gamma(x)]"] & \underline{\underline{H}}^i\underline{\underline{I}}^\bullet_x \arrow[r] & 0
\end{tikzcd}$$

Hence $eval_*$ is an isomorphism.

<u>Theorem 5.2.3</u>: $\alpha^\bullet : D^b_{fc}(X) \to D^b_{\mathscr{C}}(|X|)$ and $R^\bullet\underline{\underline{\Gamma}} : D^b_{\mathscr{C}}(|X|) \to D^b_{fc}(X)$ are equivalences of categories, and hence $\alpha^\bullet : D^b_f(X) \to D^b_{\mathscr{C}}(|X|)$ is an equivalence of categories.

<u>Proof</u>: The last statement follows from the first by theorem 3.1.3.

We can define a natural transformation $\tilde{V} : \alpha^\bullet R^\bullet\underline{\underline{\Gamma}} \to id$ by letting $\tilde{V}_{\underline{\underline{A}}^\bullet} : \alpha^\bullet R^\bullet\underline{\underline{\Gamma A}}^\bullet \to \underline{\underline{A}}^\bullet$ be the $D^b(|X|)$ map in the following commutative diagram.

$$\begin{tikzcd}
\alpha^\bullet R^\bullet\underline{\underline{\Gamma A}}^\bullet \arrow[r, "\tilde{V}_{\underline{\underline{A}}^\bullet}"] \arrow[d, equal] & \underline{\underline{A}}^\bullet \arrow[d] \\
\alpha^\bullet\underline{\underline{\Gamma}}^\bullet\underline{\underline{I}}^\bullet \arrow[r, "V_{\underline{\underline{I}}^\bullet}"] & \underline{\underline{I}}^\bullet
\end{tikzcd},$$

where $\underline{\underline{A}}^{\bullet} \to \underline{\underline{I}}^{\bullet}$ is an injective resolution. By lemma 5.2.2, then, $\tilde{\nabla}_{\underline{\underline{A}}^{\bullet}}$ is an isomorphism in $D^b_{\mathscr{C}}(|X|)$.

The transformation $i_{\underline{\underline{A}}} : \underline{\underline{\Gamma}}\alpha\underline{\underline{A}} \to \underline{\underline{A}}$ on $S(X)$ induces a natural transformation $i_{\underline{\underline{A}}^{\bullet}} : \underline{\underline{\Gamma}}^{\bullet}\alpha^{\bullet}\underline{\underline{A}}^{\bullet} \to \underline{\underline{A}}^{\bullet}$ on $K^b_{fc}(X)$, where $i_{\underline{\underline{A}}^{\bullet}}$ is an isomorphism. In particular, $\underline{\underline{\Gamma}}^{\bullet}\alpha^{\bullet}$ is exact, hence is defined on $D^b_{fc}(X)$, and we have a natural transformation of isomorphisms $i : \underline{\underline{\Gamma}}^{\bullet}\alpha^{\bullet} \to \mathrm{id}$ on $D^b_{fc}(X)$. To complete the proof, then, we need to exhibit a natural transformation of isomorphisms $\underline{\underline{\Gamma}}^{\bullet}\alpha^{\bullet} \to R^{\bullet}\underline{\underline{\Gamma}}\alpha^{\bullet}$ on $D^b_{fc}(X)$. We define such a transformation $s_{\underline{\underline{A}}^{\bullet}} : \underline{\underline{\Gamma}}^{\bullet}\alpha^{\bullet}\underline{\underline{A}}^{\bullet} \to R^{\bullet}\underline{\underline{\Gamma}}\alpha^{\bullet}\underline{\underline{A}}^{\bullet} = \underline{\underline{\Gamma}}^{\bullet}\underline{\underline{I}}^{\bullet}$ for $\alpha^{\bullet}\underline{\underline{A}}^{\bullet} \to \underline{\underline{I}}^{\bullet}$ the standard injective resolution, to simply be $\underline{\underline{\Gamma}}^{\bullet}$ applied to this resolution. $\alpha^{\bullet}\underline{\underline{\Gamma}}^{\bullet}\alpha^{\bullet}\underline{\underline{A}}^{\bullet} \to \alpha^{\bullet}\underline{\underline{\Gamma}}^{\bullet}\underline{\underline{I}}^{\bullet}$ is a quasi-isomorphism since $\alpha^{\bullet}\underline{\underline{A}}^{\bullet} \to \underline{\underline{I}}^{\bullet}$ is, and it follows from this that $s_{\underline{\underline{A}}^{\bullet}}$ is, as well, since α gives an isomorphism between the stalk of a cell sheaf on a cell and the stalk of a topological sheaf on a point in the cell.

<u>Corollary 5.2.4</u>: If $\mathscr{S}$ is a sequence $X^0 \subseteq X^1 \subseteq \dots \subseteq X^n = X$ of closed subcomplexes of a cell complex X , then we have an equivalence of categories $\alpha^{\bullet} : D^b_{\mathscr{S}}(X) \to D^b_{|\mathscr{S}|}(|X|)$ where $D^b_{|\mathscr{S}|}(|X|)$ is the full subcategory of $D^b_{\mathscr{C}}(|X|)$ consisting of objects $\underline{\underline{A}}^{\bullet}$, where $\underline{\underline{H}}^i\underline{\underline{A}}^{\bullet}$ is locally constant on strata $X^i - X^{i-1}$.

<u>Proof</u>: This follows immediately from theorem 5.2.3 and by the fact that $\underline{\underline{A}} \in S_f(X)$ is constructible if and only if $\alpha\underline{\underline{A}}$ is locally constant on strata.

<u>Remark</u>: In practice, the major application of corollary 5.2.4 is

in the case of $|X|$ being a pseudomanifold and $|\mathscr{S}|$ a stratification of $|X|$. For more details, see [GM].

Each of the functors $\underline{\underline{H}}^i$, $\otimes$, $R^\bullet\underline{\underline{Hom}}$, D , $R^\bullet\Gamma$, $R^\bullet\Gamma_c$, $\mathbb{H}^i$, $\mathbb{H}^i_c$, $R^\bullet\mathrm{Hom}$, $\underline{\underline{Ext}}^i$, Ext^i , $\tau_{\leq p}$, $\tau^{\geq p}$, r_Y , r^Y (Y locally closed), $R^\bullet f_*$, f^* , $R^\bullet f_!$, and $f^!$ (f a continuous map) are defined on $D^b(X)$ for X a locally compact Hausdorff space. If X is a pseudomanifold with stratification $\mathscr{S}$, then each of these functors except for the last four are defined when categories $D^b(X)$ are replaced with categories $D^b_{\mathscr{S}}(X)$ (to see this, and to see conditions under which $R^\bullet f_*$, f^* , $R^\bullet f_!$, and $f^!$ are defined between constructible derived categories, see Borel's paper in [B], and [GM]).

<u>Theorem 5.2.5</u>: Given X a cell complex, with $|X|$ a pseudomanifold, then each functor in the list above in the topological theory corresponds to the functor with the same name in the cellular theory by the map $\alpha^\bullet : D^b_{fc}(X) \to D^b_{fc}(|X|)$, where, with $R^\bullet f_*$, f^* and $f^!$, f must be a cellular map, and with $R^\bullet f_!$, f must be a fibred cellular map.

<u>Proof</u>: The cases $\underline{\underline{H}}^i$, $\tau_{\leq p}$, $\tau^{\geq p}$, r_Y , $\otimes$, and f^* are clear.

To check $R^\bullet f_*$ for a map $f : X \to Y$, we will show that for $\underline{\underline{A}}^\bullet \in D^b_{\mathscr{C}}(|X|)$, $R^\bullet f_* R^\bullet \underline{\underline{\Gamma}}\underline{\underline{A}}^\bullet$ and $R^\bullet \underline{\underline{\Gamma}} R^\bullet f_* \underline{\underline{A}}^\bullet$ are canonically isomorphic. The result then follows by the fact that $R^\bullet\underline{\underline{\Gamma}}$ and $\alpha^\bullet$ are inverses.

For $\underline{\underline{A}} \in S(|X|)$, $\underline{\underline{\Gamma}} f_*\underline{\underline{A}}(\sigma) = \Gamma(|st(\sigma)|, f_*\underline{\underline{A}}) = \Gamma(f^{-1}|st(\sigma)|, \underline{\underline{A}})$;

also, $f_*\underline{\underline{\Gamma A}}(\sigma) = \Gamma(f^{-1}(st(\sigma)),\underline{\underline{\Gamma A}}) =$ collections of sections $\Gamma(|st(\gamma)|,\underline{\underline{A}})$ for $\gamma \in f^{-1}(st(\sigma))$ that agree on overlaps. But these are exactly the elements of $\check{H}^0(\{|st(\gamma)|\},\underline{\underline{A}})$ for the covering $\{|st(\gamma)|\,|\gamma \in f^{-1}(st(\sigma))\}$ of $f^{-1}(|st(\sigma)|)$, which is a Leray covering (lemma 5.2.1), so $f_*\underline{\underline{\Gamma A}}(\sigma) = \check{H}^0(|f^{-1}(st(\sigma))|,\underline{\underline{A}}) = \Gamma(|f^{-1}(st(\sigma))|,\underline{\underline{A}})$. Hence $\underline{\underline{\Gamma}} f_*\underline{\underline{A}}(\sigma) = f_*\underline{\underline{\Gamma A}}(\sigma)$ for $\underline{\underline{A}} \in S(|X|)$, and in both cases the co-restriction maps and maps induced by sheaf maps $\underline{\underline{A}} \to \underline{\underline{B}}$ are the natural ones, so $\underline{\underline{\Gamma}} f_*\underline{\underline{A}}$ and $f_*\underline{\underline{\Gamma A}}$ are canonically isomorphic. For $\underline{\underline{A}}^\bullet \in K^b_{fc}(|X|)$, then, $\underline{\underline{\Gamma}}^\bullet f_*\underline{\underline{A}}^\bullet$ and $f_*\underline{\underline{\Gamma}}^\bullet \underline{\underline{A}}^\bullet$ are canonically isomorphic.

Note that for $\underline{\underline{I}}$ an injective topological sheaf on $|X|$, we have $f_*\underline{\underline{I}}$ injective, since $\underline{\underline{I}}$ being injective is equivalent to $\mathrm{Ext}^1(\underline{\underline{A}},\underline{\underline{I}}) = 0\ \forall\ \underline{\underline{A}} \in S(|X|)$, and for any $\underline{\underline{B}} \in S(|Y|)$ we have $\mathrm{Ext}^1(\underline{\underline{B}},f_*\underline{\underline{I}}) \cong \mathrm{Ext}^1(f^*\underline{\underline{A}},\underline{\underline{I}})$. We also have that $\alpha\underline{\underline{\Gamma I}}$ is injective for $\underline{\underline{I}} \in S(|X|)$ injective since a diagram $\alpha\underline{\underline{\Gamma I}} \longleftarrow \underline{\underline{A}} \hookrightarrow \underline{\underline{B}}$ in $S(|X|)$ can be factored by a map $\alpha\underline{\underline{\Gamma I}} \xrightarrow{V_{\underline{\underline{I}}}} \underline{\underline{I}} \xrightarrow{h} \underline{\underline{B}}$, and hence $\underline{\underline{\Gamma I}}$ is injective since $\alpha : S(X) \to S_{\mathscr{C}}(|X|)$ is an equivalence of categories.

Now suppose $\underline{\underline{A}}^\bullet \in D^b_{\mathscr{C}}(|X|)$ and $\underline{\underline{A}}^\bullet \to \underline{\underline{I}}^\bullet$ is an injective resolution. Then we have canonical isomorphisms $R^\bullet f_* R^\bullet \underline{\underline{\Gamma}}\,\underline{\underline{A}}^\bullet = R^\bullet f_*\underline{\underline{\Gamma}}^\bullet \underline{\underline{I}}^\bullet = f_*\underline{\underline{\Gamma}}^\bullet \underline{\underline{I}}^\bullet$ (since $\underline{\underline{\Gamma}}^\bullet \underline{\underline{I}}^\bullet$ is injective) $= \underline{\underline{\Gamma}}^\bullet f_*\underline{\underline{I}}^\bullet = R^\bullet \underline{\underline{\Gamma}} f_*\underline{\underline{I}}^\bullet$ (since $f_*\underline{\underline{I}}^\bullet$ is injective) $= R^\bullet \underline{\underline{\Gamma}} R^\bullet f_*\underline{\underline{A}}^\bullet$.

We now claim that in $D^b_{fc}(|X|)$, $\mathbb{D}\alpha([\sigma]) \overset{qi}{=} \alpha((\sigma))[\dim\sigma]$, where (σ) is $\mathbb{Q}$ on σ and 0 elsewhere. $\mathbb{D}\alpha([\sigma]) = R^\bullet \underline{\underline{\mathrm{Hom}}}(Rj_!\underline{\underline{\mathbb{Q}}},\underline{\underline{\mathbb{D}}}^\bullet_{|X|})$, where $j : |\bar{\sigma}| \hookrightarrow |X|$, $\overset{qi}{=} Rj_* R^\bullet \underline{\underline{\mathrm{Hom}}}(\underline{\underline{\mathbb{Q}}}, j^!\underline{\underline{\mathbb{D}}}^\bullet_{|X|}) \overset{qi}{=} Rj_*\underline{\underline{\mathbb{D}}}^\bullet_{|\bar{\sigma}|}$. But

$\underline{\underline{H}}^{-i}(|\bar\sigma|, \underline{\underline{\mathbb{D}}}^{\bullet}_{|\bar\sigma|})_x \cong H_i(|\bar\sigma|, |\bar\sigma| - \{x\}; \mathbb{Q})$, which is $\mathbb{Q}$ for $x \in \sigma$ and $i = \dim\sigma$, and 0 otherwise. Then $\underline{\underline{\mathbb{D}}}^{\bullet}_{|\bar\sigma|} \overset{qi}{=} \alpha((\sigma))[\dim\sigma]$ on $|\bar\sigma|$, and hence $D\alpha([\sigma]) \overset{qi}{=} Rj_*\underline{\underline{\mathbb{D}}}^{\bullet}_{|\bar\sigma|} \overset{qi}{=} \alpha((\sigma))[\dim\sigma]$ on $|X|$.

To show the result for $R^{\bullet}\underline{\underline{Hom}}$, it suffices to show that $R^{\bullet}\underline{\underline{Hom}}(\alpha^{\bullet}[\sigma],\alpha^{\bullet}[\tau])$ and $\alpha^{\bullet}R^{\bullet}\underline{\underline{Hom}}([\sigma],[\tau])$ are canonically isomorphic. The functors $\underline{\underline{Hom}}^{\bullet}$ agree on elementary injectives in a canonical way, so we need to show that $\alpha^{\bullet}[\tau]$ is acyclic with respect to $R^{\bullet}\underline{\underline{Hom}}(\alpha^{\bullet}[\sigma],\cdot)$, i.e., $\underline{\underline{Ext}}^i(\alpha^{\bullet}[\sigma],\alpha^{\bullet}[\tau]) = 0$ for $i > 0$. But $R^{\bullet}\underline{\underline{Hom}}(\alpha^{\bullet}[\sigma],\alpha^{\bullet}[\tau]) \overset{qi}{=} D(\alpha^{\bullet}[\sigma] \otimes D\alpha^{\bullet}[\tau]) \overset{qi}{=} D(\alpha^{\bullet}([\sigma]) \otimes \alpha^{\bullet}((\tau))[\dim\tau])$ $\overset{qi}{=} D(\alpha^{\bullet}((\tau))[\dim\tau])$ if $\tau \le \sigma$ and 0 otherwise. For $\tau \le \sigma$, this is $DD\alpha^{\bullet}([\tau]) \overset{qi}{=} \alpha^{\bullet}([\tau])$, so either way, $\underline{\underline{Ext}}^i(\alpha^{\bullet}[\sigma],\alpha^{\bullet}[\tau]) = 0$ for $i \neq 0$.

To prove the theorem for $j^!$ where $j : Y \hookrightarrow X$ is the inclusion of a closed subcomplex, we note that in both categories, $j^!\underline{\underline{A}}^{\bullet} \overset{qi}{=} j^*Rj_*j^!\underline{\underline{A}}^{\bullet} \overset{qi}{=} j^*Rj_*R^{\bullet}\underline{\underline{Hom}}(\underline{\underline{\mathbb{Q}}},j^!\underline{\underline{A}}^{\bullet}) \overset{qi}{=} j^*R^{\bullet}\underline{\underline{Hom}}(Rj_!\underline{\underline{\mathbb{Q}}},\underline{\underline{A}}^{\bullet})$. $R^{\bullet}\underline{\underline{Hom}}$ and j^* correspond under $\alpha^{\bullet}$, and $Rj_!$ does as well since $j_!$ in each case is extension by zero and is exact (so $Rj_! = j_!$), so it follows that the functors $j^!$ correspond.

We now show that $\underline{\underline{\mathbb{D}}}^{\bullet}_{|X|} \overset{qi}{=} \alpha^{\bullet}\underline{\underline{D}}^{\bullet}_X$. If $j : |\bar\sigma| \hookrightarrow |X|$ is the inclusion of a closed cell, then $j^*D\alpha^{\bullet}\underline{\underline{\mathbb{D}}}^{\bullet}_X \overset{qi}{=} Dj^!\alpha^{\bullet}\underline{\underline{\mathbb{D}}}^{\bullet}_X \overset{qi}{=} D\alpha^{\bullet}j^!\underline{\underline{\mathbb{D}}}^{\bullet}_X \overset{qi}{=} D\alpha^{\bullet}\underline{\underline{\mathbb{D}}}^{\bullet}_{\bar\sigma}$ $\overset{qi}{=} D(\alpha^{\bullet}((\sigma)[\dim\sigma])) \overset{qi}{=} D(D\alpha^{\bullet}([\sigma])) \overset{qi}{=} \alpha^{\bullet}[\sigma]$. Then $D\alpha\underline{\underline{\mathbb{D}}}^{\bullet}_X$ is a local system, hence $\alpha\underline{\underline{\mathbb{D}}}^{\bullet}_X = D\underline{\underline{L}}$ for a local system $\underline{\underline{L}}$ on $|X|$.

We have maps $\underline{\underline{\mathbb{Q}}}_{|X|} \to \underline{\underline{\mathbb{D}}}^{\bullet}_{|X|}$ and $\underline{\underline{\mathbb{Q}}}_{|X|} = \alpha\underline{\underline{\mathbb{Q}}}_X \to \alpha\underline{\underline{\mathbb{D}}}^{\bullet}_X$ which are quasi-isomorphisms on the non-singular part $|X| - \Sigma$ of $|X|$, so there exists a quasi-isomorphism $i^*\alpha^\bullet \underline{\underline{\mathbb{D}}}^\bullet_X \to i^*\underline{\underline{\mathbb{D}}}^\bullet_{|X|}$ for $i : |X|-\Sigma \hookrightarrow |X|$. We then have a quasi-isomorphism $Di^*\underline{\underline{\mathbb{D}}}^\bullet_{|X|} \to Di^*\alpha^\bullet\underline{\underline{\mathbb{D}}}^\bullet_X$ or $i^*\underline{\underline{\mathbb{Q}}} \to i^*\underline{\underline{L}}$, since i is an open inclusion. But a quasi-isomorphism between two single sheaves is simply a sheaf map which is an isomorphism, so we have that $\underline{\underline{L}}$ is the constant sheaf on $|X| - \Sigma$. Since $|X|$ is a pseudomanifold, $\underline{\underline{L}}$ must then be the constant sheaf on $|X|$. Then $\alpha^\bullet\underline{\underline{\mathbb{D}}}^\bullet_X \overset{qi}{=\!=} D\underline{\underline{L}} \overset{qi}{=\!=} D\underline{\underline{\mathbb{Q}}} \overset{qi}{=\!=} \underline{\underline{\mathbb{D}}}^\bullet_{|X|}$.

The rest of the functors correspond because of the following identities: $R^\bullet f_! = DR^\bullet f_* D$, $f^! = Df^*D$, $r^Y = Dr_Y D$, $R^\bullet\Gamma = R^\bullet f_*$ for $f : X \to pt.$, $R^\bullet\Gamma_c = R^\bullet f_!$ (same f), $\mathbb{H}^\bullet = \underline{\underline{H}}^\bullet R^\bullet\Gamma$, $\mathbb{H}^\bullet_c = \underline{\underline{H}}^\bullet R^\bullet \Gamma_c$, $R^\bullet Hom = R^\bullet\Gamma R^\bullet \underline{\underline{Hom}}$, $\underline{\underline{Ext}}^\bullet = \underline{\underline{H}}^\bullet R^\bullet \underline{\underline{Hom}}$, and $Ext^\bullet = \mathbb{H}^\bullet R^\bullet \underline{\underline{Hom}}$.

<u>Corollary 5.2.6</u>: In the topological derived category theory, each functor in theorem 5.2.5 is still defined when categories $D^b_{fc}(\mathfrak{X})$ are replaced with categories $D^b_{\mathscr{C}}(|X|)$ where X is a cell complex with $|X| = \mathfrak{X}$, a pseudomanifold.

<u>Proof</u>: This is an immediate consequence of theorems 5.2.3 and 5.2.5.

<u>Remark</u>: It also follows from theorems 5.2.3 and 5.2.5 that all functors of theorem 5.2.5 except for f^*, $R^\bullet f_*$, $f^!$, $R^\bullet f_!$ are defined on the cellular category $D^b_{\mathscr{S}}(X)$ if $|\mathscr{S}|$ is a stratification of a pseudomanifold $|X|$, since these functors are defined on $D^b_{|\mathscr{S}|}(|X|)$. In this case, $f^!$ is defined if f maps each stratum into a stratum

since $f^! = Df^*D$. As for $R^\bullet f_*$ and $R^\bullet f_!$, these are defined between categories $D^b_{\mathscr{S}}(X)$ any time their underlying topological functors are defined between categories $D^b_{|\mathscr{S}|}(|X|)$. This is the case for any f for which $|f|$ is <u>stratified</u>, i.e.,

(i) the inverse image of any connected component of a stratum is a union of connected components of strata;

(ii) given $|X^i| - |X^{i-1}|$ a stratum and a point $x \in |X^i| - |X^{i-1}|$ there exists a neighborhood N of x in $|X^i|$, a topologically stratified space $F = F^k \supseteq F^{k-1} \supseteq \ldots \supseteq F^0$ and a stratum preserving homeomorphism $F \times N \to |f|^{-1}(N)$ which commutes with the projection to N.

See [GM] for more details.

REFERENCES

[B] A. Borel, et al.: Seminar on Intersection Homology, Bern, Switzerland, to appear in Springer Lecture Notes in Mathematics.

[BM] A. Borel and J. Moore, Homology Theory for Locally Compact Spaces, Michigan Math J. 7 (1960), pp.137-159

[CF] G. Cooke and R. Finney, Homology of Cell Complexes, Princeton University Press, Princeton, NJ (1967)

[GM] M. Goresky and R. MacPherson, Intersection Homology II, Inv. Math. 72 (1983), pp. 77-130

[H] R. Hartshorn, Residues and Duality, Springer Lecture Notes in Mathematics 20 (1966)

[Iv] B. Iverson, Cohomology of Sheaves, preprint, Aarhus, Denmark (1976)

[K] M. Kashiwara, The Riemann-Hilbert Problem, preprint, R.I.M.S., Kyoto University (1983)

[V] J. Verdier, Categories Derivees, Etat 0. SGA 4½, Springer Lecture Notes in Mathematics 569 (1977)