METRICS & STRATIFICATION THEORY IN MODERN APPLIED TOPOLOGY

A TALK BY

Asst. Prof. JUSTIN CURRY

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HISTORY

- TOPOLOGY IS THE YOUNGEST BRANCH OF MATHEMATICS ⇝ Riemann 1850s
  cf. ALGEBRA, GEOMETRY, ANALYSIS, LOGIC
  Betti 1870s
  Poincaré ~1900

- SINGULARITY THEORY WAS ONE OF THE FIRST APPLICATIONS OF DIFFERENTIAL TOPOLOGY ⇝ Maxwell (1850s), Morse (1920s), Whitney (1950s), Thom (1960s), Mather (1970s) {Arnold (1980s)

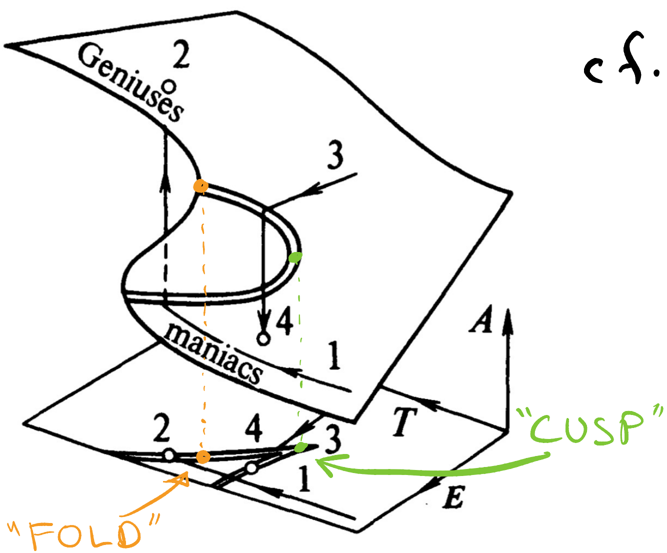
  ↳ $f: M \to N$ smooth mapping

  study where $Df: TM \to TN$ is not full rank as well as "normal forms" of $Df(p)$

CATASTROPHE THEORY

- FIRST REVOLUTION OF APPLIED TOPOLOGY
- ENDED IN DISGRACE by late 70s

⇝ Sussman & Zahler 1977, Smale 1978



cf. ARNOLD's parody of an application of WHITNEY's CUSP & FOLD CLASSIFICATION by ZEEMAN.

A = Achievement
T = Technical Proficiency
E = Enthusiasm

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LESSONS LEARNED

On the other hand around CT, not only does mathematics come first but one sees a sort of mathematical egocentricity; understanding the world is a mathematical (even geometrical) problem. Thom's position on this is clear; e.g. "... Eliminate the "obvious" meaning and replace it by the purely abstract geometrical manipulation of forms. The only possible theoretisation is Mathematical." (638, ZCT) or "... I agree with P. Antonelli, when he states that theoretical biology should be done in Mathematical Departments; we have to let biologists busy themselves with their very concrete–but almost meaningless–experiments; in developmental Biology, how could they hope to solve a problem they cannot even formulate?" (636, ZCT).

BE HUMBLE

USE NUMBERS / COLLECT DATA

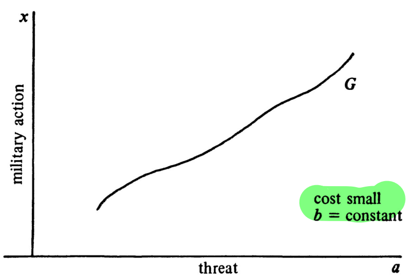


FIGURE 6, 316, ZCT

? ⇒

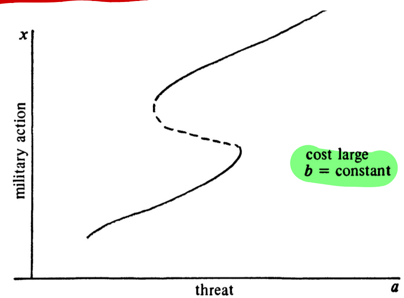
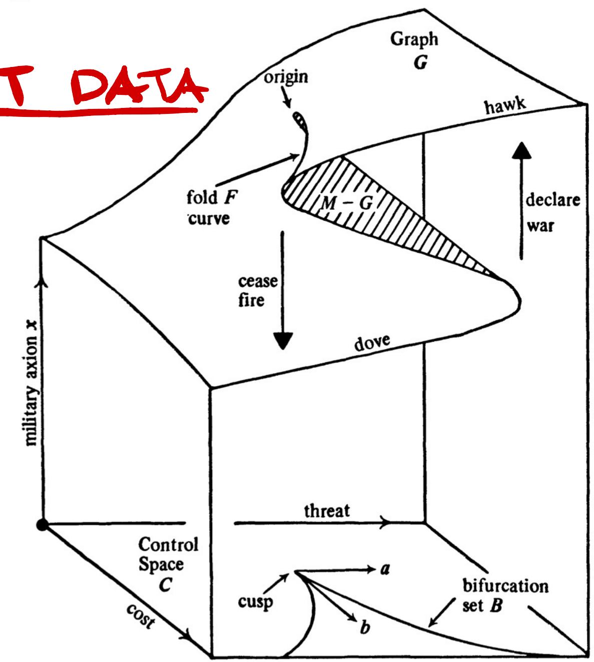


FIGURE 7, 317, ZCT

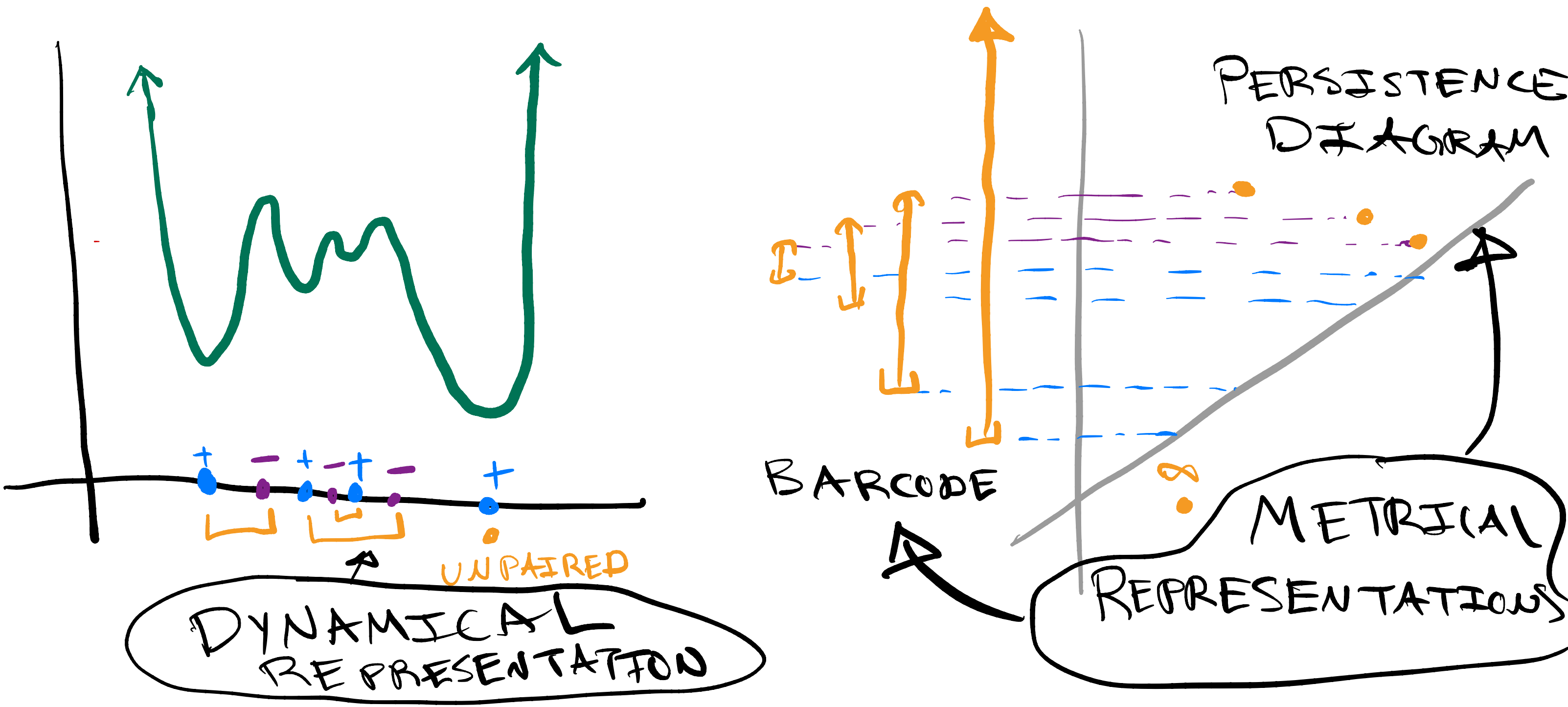


One trouble is the definition for "small" cost and "large" cost that Zeeman and Isnard use. They mean there exists b_0 such that "small b" means $b < b_0$ and "large b" means $b > b_0$, the same b_0 for "small" and "large". Thus Figures 6 and 7 (the sociological hypotheses) already describe the model for all b save b_0; Figure 6 applies if $b < b_0$, Figure 7 if $b > b_0$ [p. 331, ZCT]. No evidence or justification is given for this sociological hypothesis, that such a b_0 exists (there are arguments given earlier however to justify a "delay rule" vs. "Maxwell's rule").

DO STATISTICS!

Topological Data Analysis: Catastrophe Theory Redux?

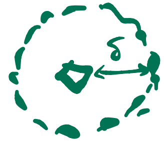
- Uses COMPUTERS and tries to empower users
- Incorporates METRIC information, e.g. SCALE



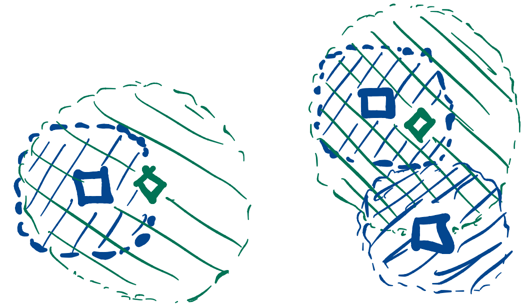
INTERLEAVING DISTANCE

- GENERALIZATION OF THE HAUSDORFF DISTANCE

$X = \square \rightsquigarrow X^{\varepsilon} =$ 

$Y = \diamond \rightsquigarrow Y^{\delta} =$ 

- FIND ε s.t. $Y \subset X^{\varepsilon}$
- FIND δ s.t. $X \subset Y^{\delta}$
- CHOOSE $\varepsilon' := \max(\varepsilon, \delta)$



$$\begin{tikzcd} & & X^{\varepsilon+\delta} & & Y^{\delta+\varepsilon} \\ & & X^{\varepsilon} \arrow[u] \arrow[urr] & & Y^{\delta} \arrow[u] \arrow[ull] \\ & & X \arrow[u] \arrow[urr] & & Y \arrow[u] \arrow[ull] \end{tikzcd}$$

Inclusion of sets & their thickenings

ε'-INTERLEAVING DIAGRAM

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X^{\varepsilon'} \arrow[r] \arrow[dr] & X^{2\varepsilon'} \\ Y \arrow[r] \arrow[ur] & Y^{\varepsilon'} \arrow[r] \arrow[ur] & Y^{2\varepsilon'} \end{tikzcd}$$

HAUSDORFF STABILITY

~ 2005 Cohen-Steiner, et al

- $d_H(X,Y) = \inf\{\varepsilon' \mid \exists\, \varepsilon' \text{ interleaving}\}$

Here we view X, Y as functors from $\underline{\mathbb{R}_{\geq 0}}$ to $\underbrace{\text{subsets of } \mathbb{R}^2 \text{ w/ inclusion}}_{\mathcal{C}}$ or $\underline{\text{TOP}}$

top. spaces & inclusion

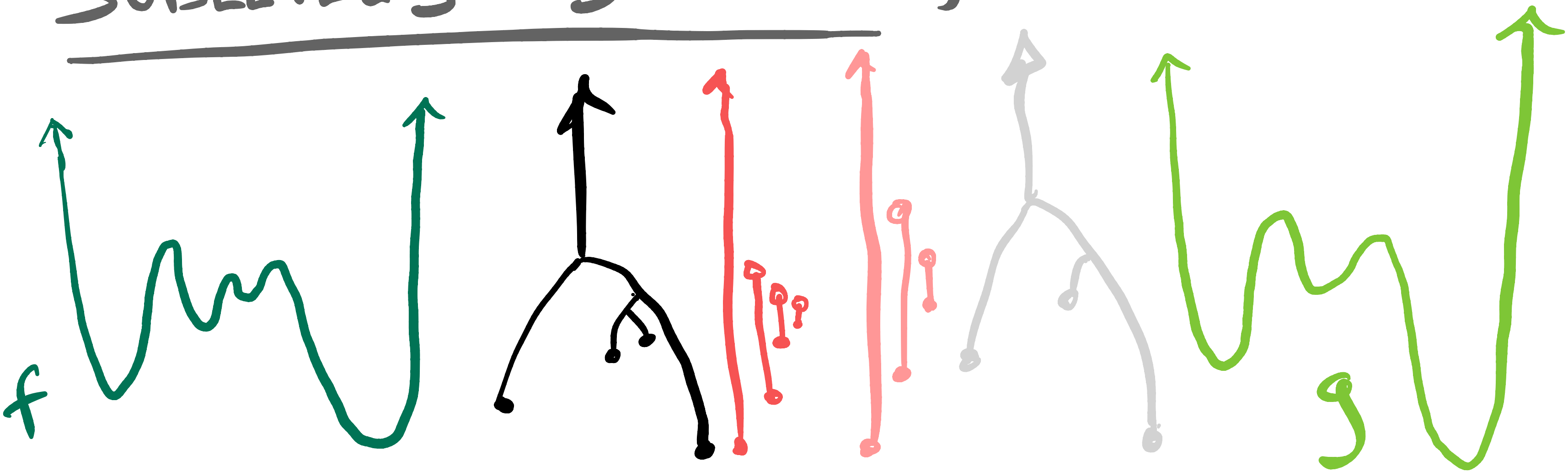
- Of course, applying Homology, provides a completely analogous definition

$$d_I(H_nX, H_nY) = \inf_{\varepsilon} \text{ s.t. } \left\{ \begin{tikzcd} H_nX \arrow[r] \arrow[dr] & H_nX^{\varepsilon} \arrow[r] \arrow[dr] & H_nX^{2\varepsilon} \cdots \\ H_nY \arrow[r] \arrow[ur] & H_nY^{\varepsilon} \arrow[r] \arrow[ur] & H_nY^{2\varepsilon} \cdots \end{tikzcd} \right.$$

THM For $n \geq 0$

$$d_I(H_nX, H_nY) \leq d_H(X,Y)$$

SUBLEVEL SET STABILITY



$F : \mathbb{R} \to Top$
$t \longmapsto f^{-1}(-\infty, t]$

$G : \mathbb{R} \to Top$
$t \longmapsto g^{-1}(-\infty, t]$

Def: ε-shift $F^{\varepsilon}(t) = F(t+\varepsilon)$
$= f^{-1}(-\infty, t+\varepsilon]$

N.B. $d_I(F,G) \leq \|f-g\|_\infty$

and $H_n : Top \to Vect$ } $\pi_0 : Top \to Set$ are LIPSCHITZ.

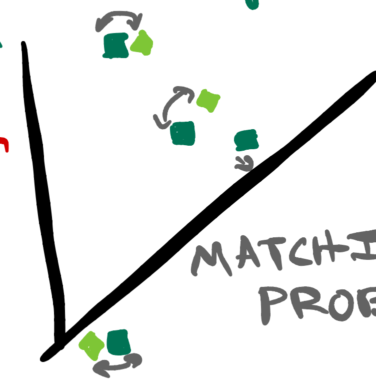
THMS $d_I(H_nF, H_nG)$ } $d_I(\pi_0 F, \pi_0 G) \leq \|f-g\|_\infty$

COMPUTABILITY

- IN GENERAL, d_I is NP HARD to compute, but

LESNICK'S ISOMETRY THM: $d_I(H_n F, H_n G) = d_B(H_n F, H_n G)$

where d_B = BOTTLENECK DIST., which is computable using the HUNGARIAN ALGORITHM



MATCHING PROBLEM

- EVEN MERGE TREE INTERLEAVING DISTANCE IS HARD.

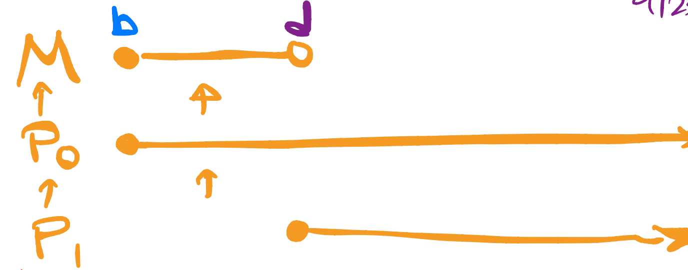
- WOULD BE NICE TO LEVERAGE $\pi_0 \longrightarrow H_0$ MAP, BUT EXPONENTIAL INVERSE PROBLEMS

$T_1 \not\cong T_2$ BUT $H_0 T_1 \cong H_0 T_2$

$\Rightarrow$ NONTRIVIAL FIBERS

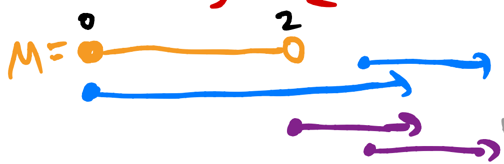
(cf. CURRY "FIBER")

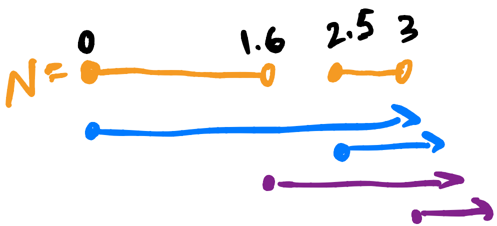
PRESENTATIONS



- Consider projective resolution
- Elementary projectives are "born" in a single graded degree.
- By inserting redundant generators & relations we can find compatible presentations for M & N assuming $\varprojlim M \to \varinjlim M \cong \varprojlim N \to \varinjlim N$

EX:





$$\begin{array}{c|cc} & F_2 & F_{2.75} \\ \hline F_0 & 1 & 0 \\ F_{2.75} & 0 & 1 \end{array}$$

$$\begin{array}{c|cc} & F_{1.6} & F_3 \\ \hline F_0 & 1 & 0 \\ F_{2.5} & 0 & 1 \end{array}$$

- Use to define ℓ^P dist!

LABEL VECTORS

$L_M = [0, 2.75; 2, 2.75]$

$L_N = [0, 2.5; 1.6, 3]$

$\Longrightarrow d_I^2 = \sqrt{(1/4)^2 + (.4)^2 + (1/4)^2}$

PRESENTATIONS FOR MERGE TREES

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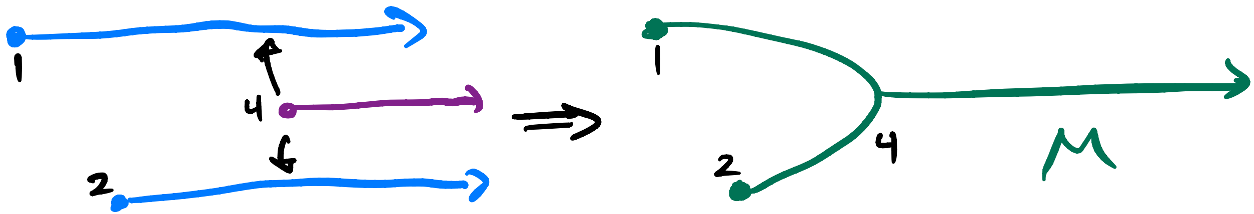
- GENERAL PHILOSOPHY: REPLACE COKERS w/ COEQUALIZERS



- PRESENTATION DISTANCE

$$\hat{d}_I^p(M,N) := \inf\left\{ \|L(P_M) - L(P_N)\|_p \text{ s.t. } P_M \,\&\, P_N \text{ compatible} \right\}$$

FAILS Δ-Ineq BUT "UNIVERSAL" REPLACEMENT

$$d_I^p(M,N) := \inf\left\{ \sum_{i=0}^{n-1} \hat{d}_I^p(M_i, M_{i+1}) \;\middle|\; M_0 = M \quad M_n = N \right\}$$

- WASSERSTEIN STABILITY (for MTs w/ Cardona & Lam & Lesnick)

$$d_{W^p}(H_0 M, H_0 N) \le d_I^p(M,N)$$

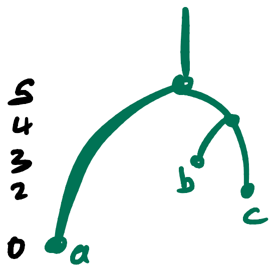
INCLUDE ℓ^p PENALTIES

also for pers. modules

$$d_{W^p}(H_n F, H_n G) \le d_I^p(F,G)$$

BJERKVIK & LESNICK

COUNTEREXAMPLE TO COPHENETIC STABILITY



$$\Longrightarrow \quad \begin{array}{c|ccc} & a & b & c \\ \hline a & & & \\ b & & & \\ c & & & \end{array} \begin{array}{l} \\ \left[\begin{matrix} 0 & 5 & 5 \\ 5 & 3 & 4 \\ 5 & 4 & 2 \end{matrix}\right] = C_M \end{array}$$

LEAST COMMON ANCESTOR MATRIX

- Munch & Stefanou '17: For ordered MTs w/ same # leaf nodes

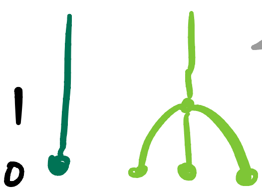
$$d_I(M,N) = \|C_M - C_N\|_\infty$$

- Gasparovic, Munch, Oudot, Turner, B. Wang, Y. Wang '19

$$d_I(M,N) = \inf_{\pi:\Lambda\to L} \|C_M - C_N\|_\infty$$

This is the NP part!

- Although $\|C_M - C_N\|_p$ makes sense, it is unstable



BOTH FILTRATIONS OF  → ℓ^p distance $\ell\sqrt{2}$

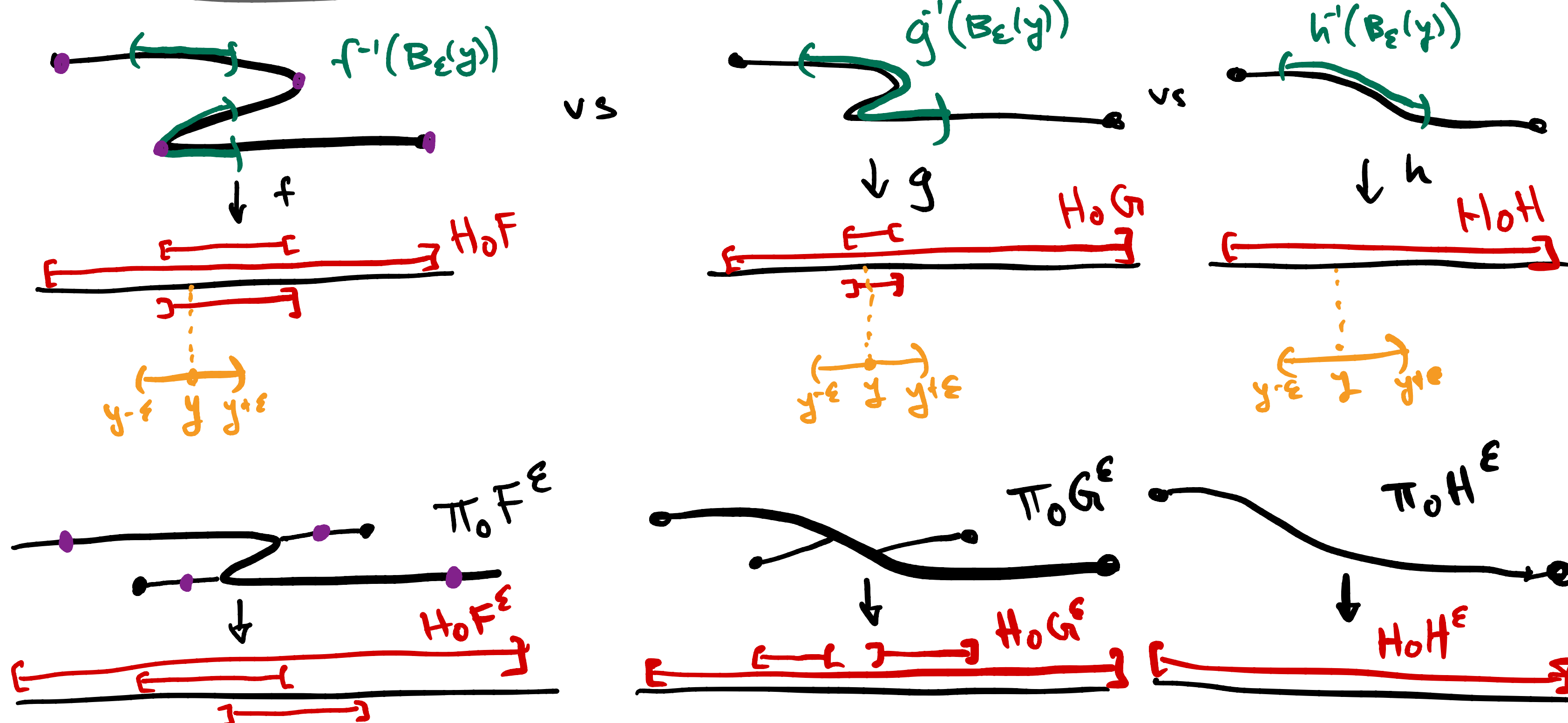
$$\left\| \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \\ & 0 & 1 \\ & & 0 \end{bmatrix} \right\|_2 = \sqrt{3}$$

BUT

$$\| [0,0,0;0,0] - [0,0,0;1,1] \|_2 = \sqrt{2}$$

QUADRATIC vs. LINEAR!

LEVEL SET STABILITY (Revist our catastrophe)



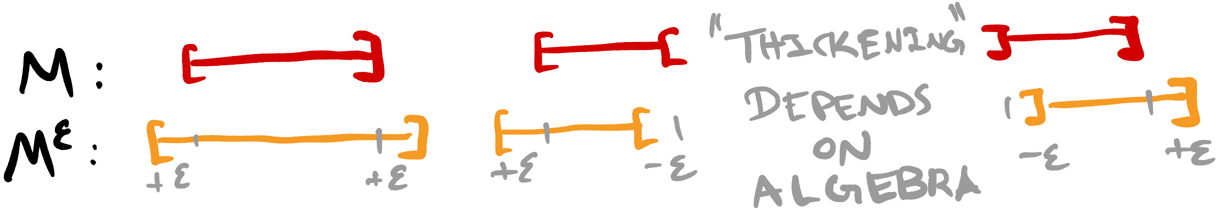
Def $F: \mathrm{Open}(\mathbb{R}) \to \mathrm{Top}$ $\quad F(U) = f^{-1}(U)$ and $F^\varepsilon(U) = f^{-1}(U^\varepsilon)$

THM: $d_I(H_nF, H_nG) \leq d_I(\pi_nF, \pi_nG) \leq \|f-g\|_\infty$

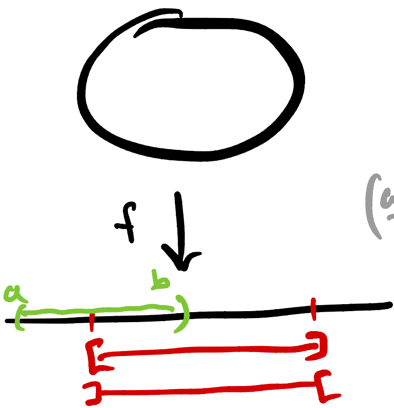
DYNAMICS ON BARCODES & THE MOVE FROM PRE-SHEAVES TO SHEAVES

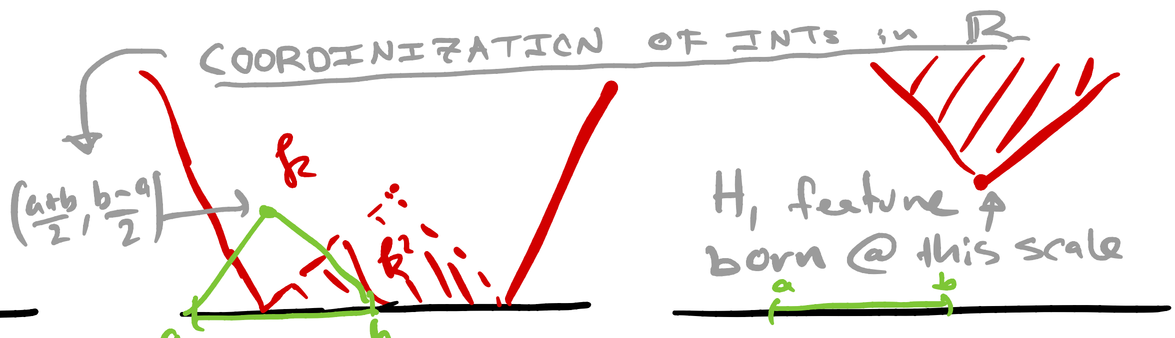
- FOUR TYPES OF INDECOMPOSABLES/BARCODES:

  CLOSED, LEFT CLOSED, RIGHT CLOSED, OPEN



- BUT THIS REPRESENTATION REQUIRES SHEAVES OTHERWISE YOU USE 2D BLOCKS!





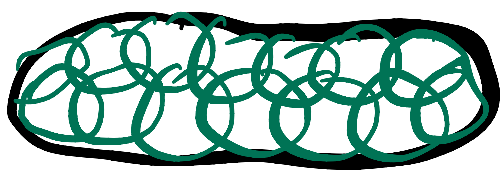
$F_0(a,b) := H_0(f^{-1}(a,b))$ $\qquad$ $F_1 := H_1(f^{-1}(a,b))$

SHEAVES & COSHEAVES

- GENERAL FUNCTORS $F : \mathrm{Open}(X)^{(op)} \to \mathcal{D}$ = DATA CATEGORY
  SET or VECT
  OR Ch(VECT)

  ARE CALLED PRE-SHEAVES (op)
  OR PRE-COSHEAES (no op)

- IN ORDER TO ENSURE LOCAL DETERMINATION
  i.e. F(U) CAN BE ASSEMBLED FROM ANY COVER OF U



U

$\mathcal{U} = \{U_i\}$
closed under intersection

SHEAF AXIOM

$$\mathcal{F}(U) \cong \lim \mathcal{F}(U_i)$$

$$\mathcal{F}(U_i) \to \mathcal{F}(U_{ij}) \leftarrow \mathcal{F}(U_j)$$

COSHEAF AXIOM

$$\mathcal{F}(U) \cong \operatorname{colim} \mathcal{F}(U_i)$$

$$\mathcal{F}(U_i) \leftarrow \mathcal{F}(U_{ij}) \to \mathcal{F}(U_j)$$

- THERE IS A PROCESS FOR FORCING THIS

☆ (CO) SHEAFIFICATION ☆

STABILITY FOR (CO)SHEAVES

- INSTEAD OF INCREASING DIMENSION, USE SHEAVES

Given $f: X \to \mathbb{R}^k$ the n^{th} Leray sheaf is

If X cpt, then $\mathcal{F}^n_x \cong H^n(f^{-1}(x))$

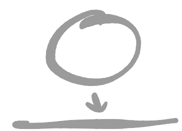
STALK @ x

$$\mathcal{F}^n = R^n f_* k_X$$ where $\mathcal{F}^n$ is the sheafification of $F^n(U) = H^n(f^{-1}(U))$

Leray Pre-Sheaf

For our example

const. sheaf on X

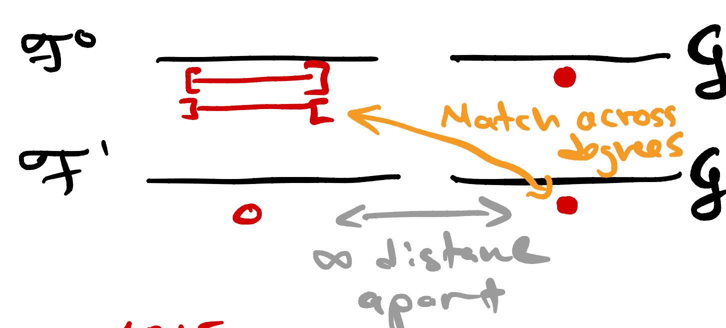


$F^1 = H^1(f^{-1}(U)) \neq 0$

BUT $\mathcal{F}^1 = 0$!

- SHEAVES ARE UNSTABLE DEGREE BY DEGREE

vs



- BUT DERIVED (and GRADED!) SHEAVES ARE STABLE

$$Rf_* k_X := f_* S^\bullet$$ where $S^\bullet$ = sheaf of singular cochains

PRESERVING LOCALITY

- EASY TO SHIFT OR "SMOOTH" PRE-COSHEAVES

$$T_\varepsilon(F) := F_n^\varepsilon(U) = H_n(f^{-1}(U^\varepsilon)) \quad \text{but} \quad \mathcal{F}_n^\varepsilon(U) = \varinjlim \overbrace{H_n(f^{-1}(B_\varepsilon(x)))}^{F_n^\varepsilon(x)}$$

cover by ε balls

Uses "small" cover, but ε is getting bigger!

- HOW TO SMOOTH COSHEAVES?

Fun(Open(X), $\mathcal{D}$) with $\mathcal{D}$ = Set or VECT or Ch(Vect)

Fun(Open(X), $\mathcal{D}$) = Pre-CoShv(X)

$$\begin{tikzcd} \mathrm{CoShv}(X) \arrow[r, bend left, "i_* \text{ = INCLUSION}"] & \text{Pre-CoShv}(X) \arrow[l, bend left, "i^* \text{ = COSHEAFIFICATION}"] \arrow[loop right, "T_\varepsilon"] \end{tikzcd}$$

T_ε: SMOOTHING ENDOFUNCTOR

$$\mathcal{F}^\varepsilon := i^* T_\varepsilon i_* \mathcal{F}$$

$$(\mathcal{F}^\varepsilon)^\varepsilon \to \mathcal{F}^{2\varepsilon} \quad \text{uses} \quad i^* T_\varepsilon \underbrace{i_* i^*}_{} T_\varepsilon i_* \Rightarrow i^* T_\varepsilon T_\varepsilon i_* \to i^* T_{2\varepsilon} i_*$$

ADJOINT PAIR $i_* \dashv i^*$

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CATEGORIES W/ A FLOW (THESIS WORK OF A. STEFANOU)

- IN GENERAL METRIC SPACES, THICKENING HAS POOR PROPERTIES: GIVEN $A \subseteq (X,d)$ $A^{\varepsilon} = \{y \mid d(A,y) \leq \varepsilon\}$
THIS DEFINES AN ACTION $T: \mathcal{P}(X) \times [0,\infty) \longrightarrow \mathcal{P}(X)$ BUT
$T_0 A \neq A$ & $T_\varepsilon T_\varepsilon A = (A^\varepsilon)^\varepsilon \subseteq A^{2\varepsilon} = T_{2\varepsilon} A$

- deSILVA, MUNCH, and STEFANOU abstracted this into a general framework using category theory.

Definition 1.2. A **category with a flow (or translation)**[1] $(\mathbf{C}, T)$ consists of a category $\mathbf{C}$, together with

- a functor $T : \mathbb{R}_{\geqslant 0} \to \mathbf{End}(\mathbf{C})$, $\epsilon \mapsto T_\epsilon$, called the **flow** or **translation**,
- a natural transformation $u : I_{\mathbf{C}} \Rightarrow T_0$, where $I_{\mathbf{C}}$ is the identity endofunctor of $\mathbf{C}$, and
- a collection of natural transformations $\mu_{\epsilon,\zeta} : T_\epsilon T_\zeta \Rightarrow T_{\epsilon+\zeta}$, $\epsilon, \zeta \geqslant 0$,

such that the diagrams

$$\begin{tikzcd} & T_\epsilon \arrow[dl, Rightarrow, "uI_{T_\epsilon}"'] \arrow[dr, equal] & \\ T_0 T_\epsilon \arrow[rr, Rightarrow, "\mu_{0,\epsilon}"] & & T_\epsilon \end{tikzcd} \qquad \begin{tikzcd} & T_\epsilon \arrow[dl, Rightarrow, "I_{T_\epsilon}u"'] \arrow[dr, equal] & \\ T_\epsilon T_0 \arrow[rr, Rightarrow, "\mu_{\epsilon,0}"] & & T_\epsilon \end{tikzcd}$$

$$\begin{tikzcd} T_\epsilon T_\zeta T_\delta \arrow[r, Rightarrow, "I_{T_\epsilon}\mu_{\zeta,\delta}"] \arrow[d, Rightarrow, "\mu_{\epsilon,\zeta}I_{T_\delta}"'] & T_\epsilon T_{\zeta+\delta} \arrow[d, Rightarrow, "\mu_{\epsilon,\zeta+\delta}"] \\ T_{\epsilon+\zeta} T_\delta \arrow[r, Rightarrow, "\mu_{\epsilon+\zeta,\delta}"'] & T_{\epsilon+\zeta+\delta} \end{tikzcd} \qquad \begin{tikzcd} T_\epsilon T_\zeta \arrow[r, Rightarrow, "\mu_{\epsilon,\zeta}"] \arrow[d, Rightarrow, "T_{(\epsilon\leqslant\delta)}T_{(\zeta\leqslant\kappa)}"] & T_{\epsilon+\zeta} \arrow[d, Rightarrow, "T_{(\epsilon+\zeta\leqslant\delta+\kappa)}"'] \\ T_\delta T_\kappa \arrow[r, Rightarrow, "\mu_{\delta,\kappa}"] & F_{\delta+\kappa} \end{tikzcd}$$

commute for every $\epsilon, \zeta, \delta, \kappa \geqslant 0$.

Definition 1.3. Let M, N be two objects in $\mathbf{C}$. A **weak ϵ-interleaving of M and N**, denoted (φ, ψ), consists of a pair of morphisms $\varphi : M \to T_\epsilon N$ and $\psi : N \to T_\epsilon M$ in $\mathbf{C}$ such that the following pentagons

$$\begin{tikzcd} T_0 M \arrow[d, "T_{(0\leqslant 2\epsilon),M}"'] & M \arrow[l, "u_M"'] \arrow[dr, "\varphi"] & N \arrow[r, "u_N"] \arrow[dl, "\psi"'] & T_0 N \arrow[d, "T_{(0\leqslant 2\epsilon),N}"] \\ & T_\epsilon M \arrow[dr, "T_\epsilon \psi"] & T_\epsilon N \arrow[dl, "T_\epsilon \phi"'] & \\ T_{2\epsilon} M & T_\epsilon T_\epsilon M \arrow[l, "\mu_{\epsilon,\epsilon,M}"'] & T_\epsilon T_\epsilon N \arrow[r, "\mu_{\epsilon,\epsilon,N}"] & T_{2\epsilon} N \end{tikzcd}$$

commute. We say that M, N are **weakly ϵ-interleaved** if there exists a weak ϵ-interleaving (φ, ψ) of M and N. The **(weak) interleaving distance** with respect to T for a pair of objects M, N in $\mathbf{C}$ is defined to be

$$d_{(\mathbf{C},T)}(M, N) = \inf\{\epsilon \geqslant 0 \mid M, N \text{ are weakly } \epsilon\text{-interleaved}\}.$$

If M and N are not weakly interleaved for any ϵ, we set $d_{(\mathbf{C},T)}(M, N) = \infty$.

PULLING BACK A FLOW

(WORK W/ MAGNUS BOTNAN & LIZ MUNCH)

10/23

cf. "A RELATIVE THEORY OF INTERLEAVINGS" (BUT THIS IS A MAJOR REWRITE)

- GIVEN A CATEGORY W/ A FLOW $(\mathcal{D}, T)$ AND AN ADJOINT PAIR $F \dashv G$

$$\tilde{T}_\varepsilon := G T_\varepsilon F \circlearrowright \mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \circlearrowleft T_\varepsilon$$

WE HAVE AN INDUCED PULLBACK FLOW $\tilde{T}$

- MOREOVER, THIS EQUIPS $\mathcal{C}$ & $\mathcal{D}$ W/ THE STRUCTURE OF LAWVERE METRIC SPACES WHERE

$$d_{(\mathcal{C},\tilde{T})}(M,N) = d_{(\mathcal{D},T)}(f_* M, f_* N)$$

IS AN ISOMETRY ONTO ITS IMAGE.

1/4/23

APPLICATION TO DISCRETIZATIONS

- SUPPOSE $f : P \to Q$ is a map of posets

e.g. $x_1 \to [x_1, x_2] \leftarrow x_2 \to [x_2, x_3] \leftarrow x_3 \to [x_3, x_4] \leftarrow x_4 = P$

$Q = \mathrm{CLOSED}(\mathbb{R})$; $Q \circlearrowleft T_\varepsilon \quad S \mapsto S^\varepsilon$

AND $Q \times \mathbb{R}_{\geq 0} \to Q$ HAS A TRANSLATION

THEN (f_*, f^*) IS AN ADJOINT PAIR

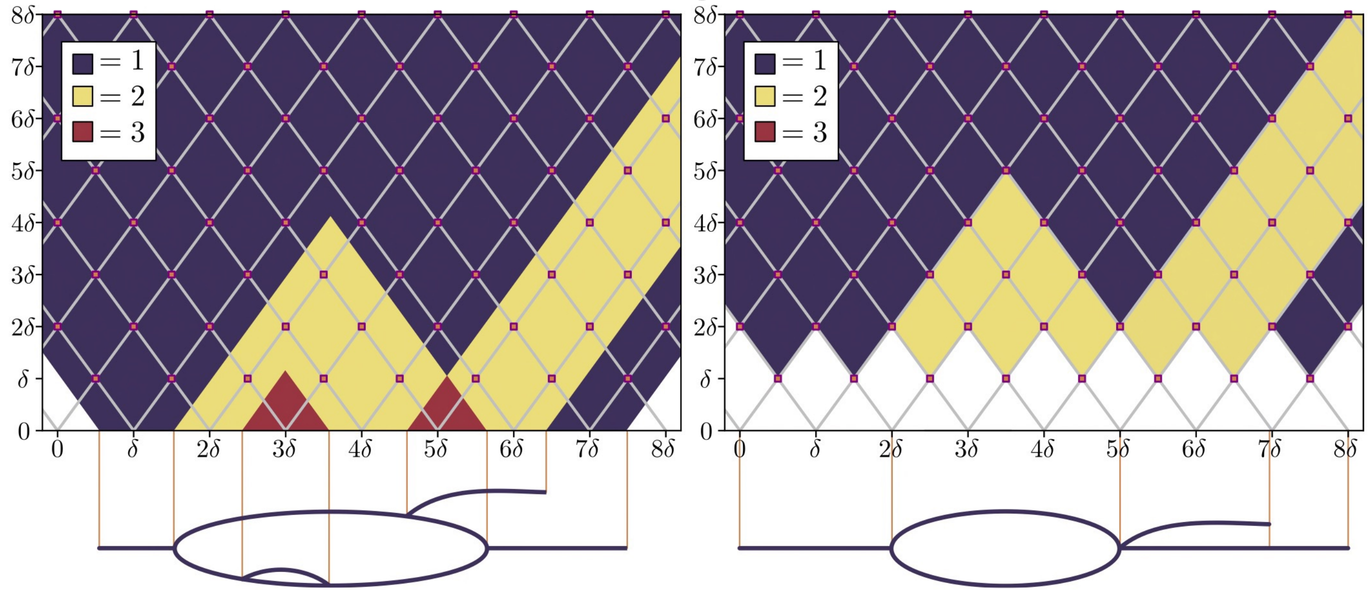
THAT ALLOWS US TO COMPUTE INTERLEAVINGS

OF $M, N \in \mathrm{FUN}(Q, \mathrm{VECT})$ WRT $f : P \to Q$

- WE THINK OF $f_* f^* M$ = "PIXELIZATION OF M"

PIXELIZATION OF REEB GRAPHS

(Figure courtesy of Liz Munch)



- PIXELIZATION ACTUALLY BREAKS THE COSHEAF AXIOM
- cf. MUNCH/WANG or KASHIWARA & SCHAPIRA

INFERENCE RESULT

f^*M

- IMAGINE $f: \mathcal{P} \to Q$ is a "SAMPLING" OF A POSET Q e.g. $\mathcal{P} =$  $Q = \mathrm{OPEN}(\mathbb{R})$
- THE Δ-INEQUALITY PROVIDES "INFERENCE"

$$d_Q(M,N) \le d_Q(M, f_*f^*M) + \underbrace{d_Q(f_*f^*M, f_*f^*N)}_{= \, d_P(f^*M, f^*N)} + d_Q(f_*f^*N, N)$$

INTRINSIC DISTANCE B/W SAMPLINGS

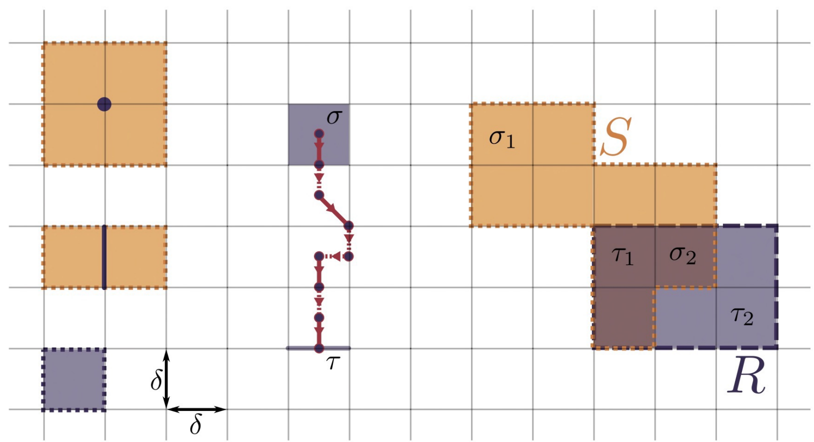
LEMMA: $|d_Q(M,N) - d_P(f^*M, f^*N)| \le d_Q(M, f_*f^*M) + d_Q(N, f_*f^*N)$

DEF: $f: \mathcal{P} \to Q$ is a δ-APPROX IF $\forall q \in Q \ \exists p \in P$ w/ $q \le f(p) \le T_\delta(q)$

THM: If $\mathcal{P}$ & Q are LATTICES & $f: \mathcal{P} \to Q$ is a δ-approx that preserves JOINS THEN WE CAN INFER INTERLEAVINGS UP TO A 2δ TOLERANCE!

WORKING LOCALLY W/ WEIGHTED POSETS

- PRACTICALLY, WE CAN REDUCE INTERLEAVINGS TO A GRID ON $\mathbb{R}^k$
- $\tilde{T}_\varepsilon$ THEN ONLY "JUMPS" AT DISCRETE VALUES OF ε ADAPTED TO THE GRID SIZE

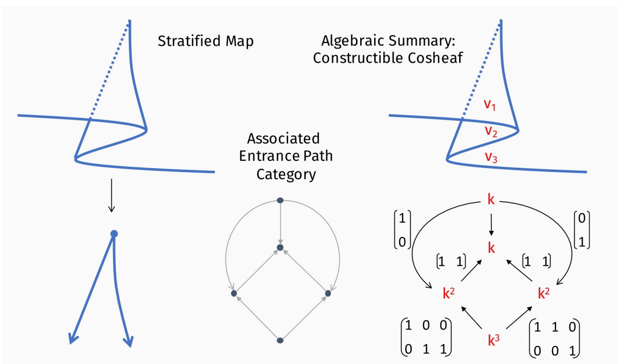


- THIS INDUCES A "WEIGHTING" ON THE FACE RELATION POSET (MONOIDAL ENRICHMENT) THAT ALLOWS US TO METRIZE CELLULAR SHEAVES & COSHEAVES

FUTURE DIRECTIONS

- THIS PROVIDES A METRIC VERSION OF MACPHERSON'S OBSERVATION

CONSTRUCTIBLE (CO)SHEAVES $\cong$ REPRESENTATIONS OF THE (ENTRANCE) EXIT PATH CATEGORY



← PREVIEW OF FRIDAYS TALK @ UF TDA 2022

AND MAKES IT EVEN MORE PERTINENT TO MODERN APPLIED TOPOLOGY!

THANK YOU!