Exemplars of Sheaf Theory in TDA

Justin M. Curry + the hard work of many others, cited in due time

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University at Albany, SUNY

First, a Word from our Sponsors...

Many thanks to the NSF for CCF Award #1850052 and NASA GRC for *TIMAEUS*.

Plan for the Talk

This talk will focus on four case studies of (co)sheaves in TDA.

Two examples *in the small*:

1. **Persistent Homology**
2. **Decorated Merge Trees**

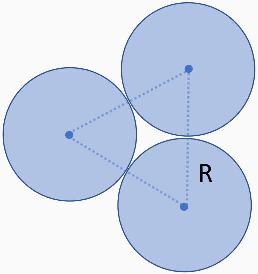
And two examples *in the large*:

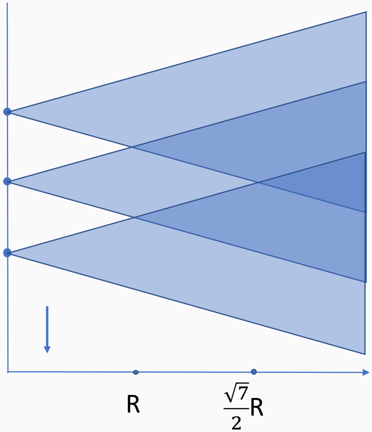
3. **The Moduli Space of Merge Trees**
4. **The Persistent Homology Transform Sheaf**

Formalizing Persistent Homology

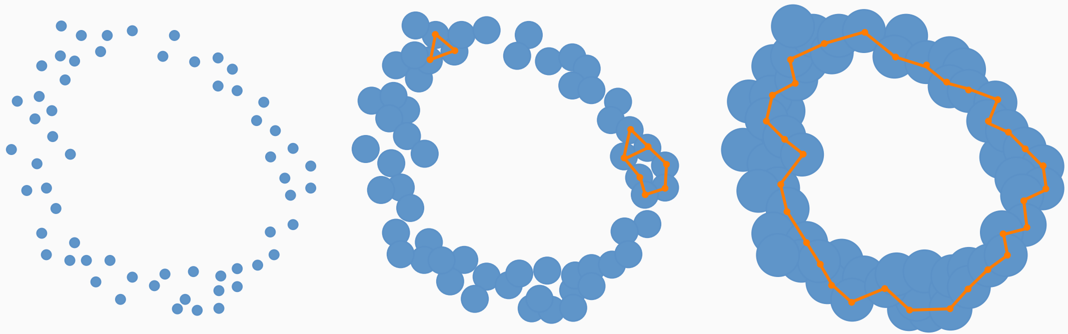
The "Algebraic Geometry" of TDA

TDA studies *maps to metric spaces* $f : X \to S$ by algebratizing $f^{-1}(s)$.



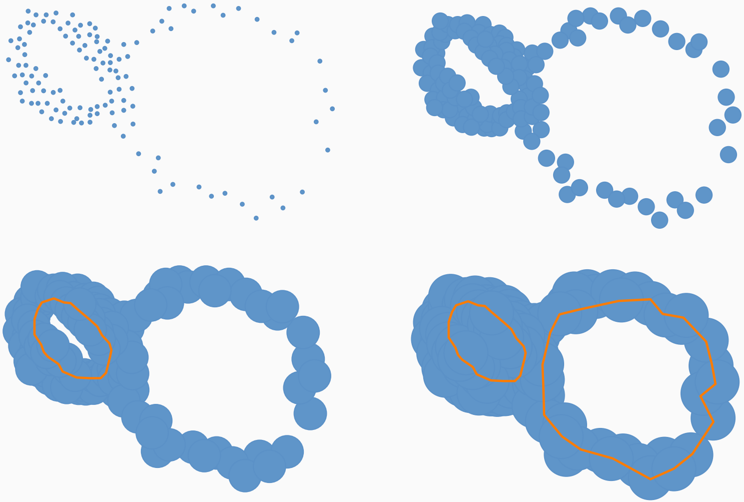


Point Cloud and Sub-level Set Filtrations



The Beauty of Functoriality

Persistent Betti numbers $\beta_i(t) := \dim H_i(F(t))$ don't track features correctly!



Summary of Multi-Parameter Persistence

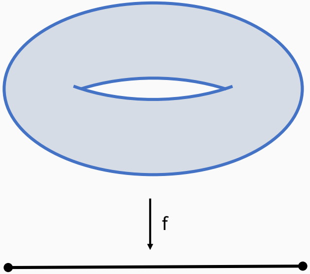
Mantra for Traditional Persistence

Study inclusions from "lower" to "higher" parameters via functoriality.

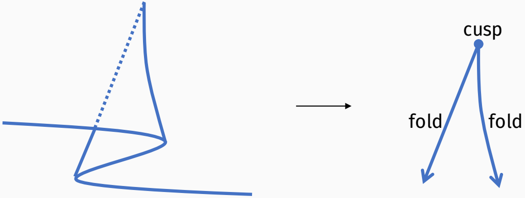
Implement, refine and apply.

Challenges of Level Set Persistence

How to organize the homology of the fibers $f^{-1}(s)$ for a general map $f : X \to S$?



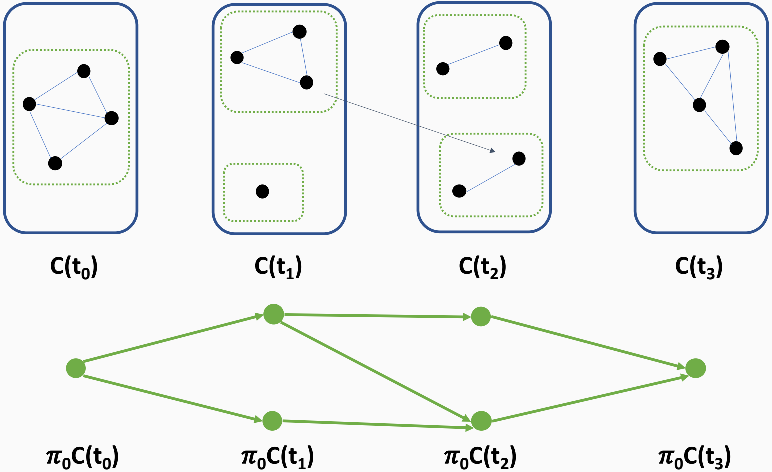
Bott's Morse function



Whitney's Cusp and Folds

Challenges of Time-Evolving Persistence

What if we wanted to study a time-evolving point cloud or network?



The Answer

- Sheaves and Cosheaves provide a unifying language for:
  - Merge Trees $\leftrightarrow$ $\mathbf{Coshv}(\mathbb{R}_{\mathsf{Alex}};\mathbf{Set})$ and Reeb graphs $\leftrightarrow$ $\mathbf{Coshv}(\mathbb{R}_{\mathsf{Eucl}};\mathbf{Set})$
  - Sublevel Set Persistence $\leftrightarrow$ $\mathbf{Shv}(\mathbb{R}_{\mathsf{Alex}})$
  - Level Set Persistence $\leftrightarrow$ $\mathbf{Shv}(\mathbb{R}_{\mathsf{Eucl}})$
  - Time-varying Persistence $\leftrightarrow$ $\mathbf{Shv}(\mathbb{R}_{\mathsf{Eucl}} \times \mathbb{R}_{\mathsf{Alex}})$
- Cosheaves serve as a *calculus of TDA* in the sense that
  - there is an existing toolbox of results and theory, and
  - one can write down analytical models and use sheaves to perform by-hand calculations.

Enter the Cosheaf

Pre-Cosheaf and Cosheaf

Let X be a topological space and $\mathbf{D}$ a category, e.g. **Set** or **Vect** A **pre-cosheaf** F

- assigns to every open set $U \subseteq X$ an object $F(U)$
- assigns to every pair $U \subseteq V$ a morphism $F(U) \to F(V)$

If the object $F(U)$ can be determined as a colimit of objects assigned to elements of any cover $\{U_i\}$ of U, then F is a **cosheaf**.

Pre-Sheaves and Sheaves

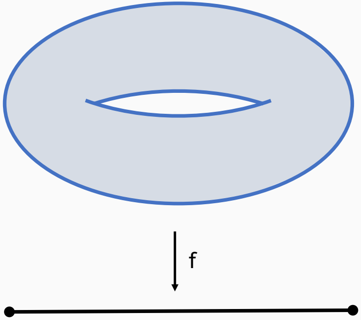
By turning around the arrows, so that F restricts from larger open sets to smaller ones, one obtains the notion of a pre-sheaf and sheaf.

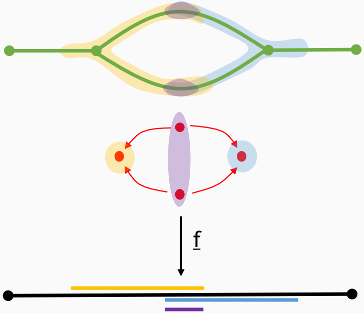
Fundamental Example I: Reeb Cosheaf

Given a map $f : Y \to X$, the **Reeb cosheaf**, which is *the* fundamental cosheaf,

$$\mathcal{R}_f : U \rightsquigarrow \pi_0(f^{-1}(U))$$

tracks components and plays the same role as the sheaf of sections does in sheaf theory.



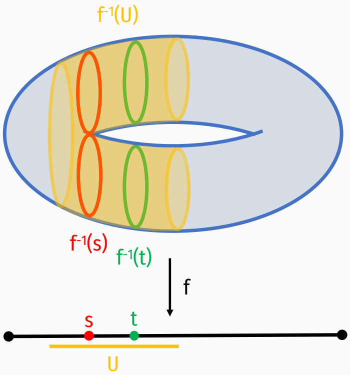


Fundamental Example II: Leray Sheaves

To a map $f : X \to Y$ the **Leray sheaf in degree** n is the sheafification of the pre-sheaf

$$F^n : U \rightsquigarrow H^n(f^{-1}(U); \mathbb{k}),$$

written as $\mathcal{F}^n$ or $R^n f_* \mathbb{k}_X$; the n^{th} right derived pushforward of the constant sheaf.



Space Level

$$f^{-1}(s) \xrightarrow{\approx} f^{-1}(U) \longleftarrow f^{-1}(t)$$

Cohomology Level

Leray sheaf in deg 1

$$k^2 \xleftarrow{\approx} k^2 \longrightarrow k^2$$

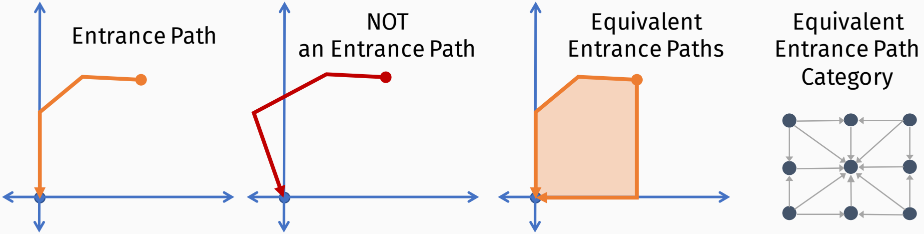
Leray sheaf in deg 0

$$k^1 \xleftarrow{\approx} k^1 \longrightarrow k^2$$

Critical fibers have quasi-isomorphic neighborhoods
that include nearby non-critical fibers

MacPherson's Entrance Path Category

Let X be stratified, i.e. partitioned into manifolds called strata. An **entrance path** is a path that only leaves a stratum by entering a lower dimensional one. Two entrance paths are **equivalent** if they are homotopic, rel endpoints, through entrance paths. The **entrance path category** $\mathbf{Ent}(X)$ has points of X for objects and equivalence classes of entrance paths for morphisms.



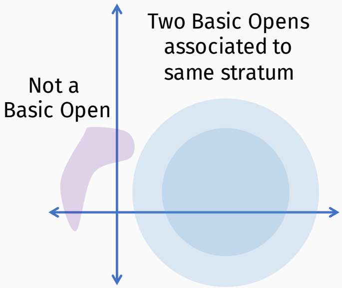
For cell complexes the entrance path category is equivalent to the face relation poset.

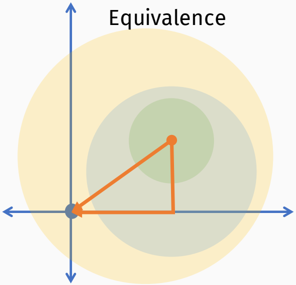
Classification of Constructible Cosheaves

Cosheaf $F : \mathbf{Open}(X) \to \mathbf{D}$ is **constructible** if whenever $U \subseteq V$ are basic opens associated to the same stratum, the morphism $F(U) \to F(V)$ is invertible.

Theorem (w/ Amit Patel)

$$\mathbf{Coshv}_c(X; \mathbf{D}) \simeq [\mathbf{Ent}(X); \mathbf{D}]$$





Constructible Cosheaves are Finite Descriptors

Corollary

If $X = \mathbb{R}$ is stratified into finitely many pieces, the entrance path category is equivalent to

$$(-\infty, t_1) \to \{t_1\} \leftarrow \cdots \to \{t_n\} \leftarrow (t_n, \infty)$$

and thus the study of constructible cosheaves is equivalent to the study of zig-zag diagrams:

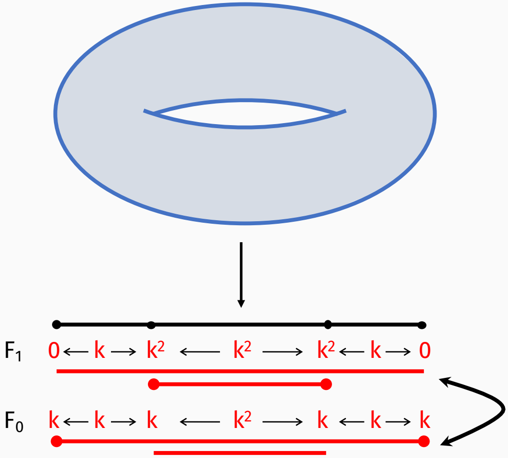
$$F(-\infty, t_1) \to F(t_1) \leftarrow \cdots \to F(t_n) \leftarrow F(t_n, \infty)$$

For $\mathbf{D} = \mathbf{Set}$, this says constructible cosheaves are equivalent to Reeb graphs.

For $\mathbf{D} = \mathbf{Vect}$, this says that constructible cosheaves are equivalent to representations of an A_{2n+1}-type quiver, which has a **barcode decomposition**.

Bott's Torus Revisited

Leray cosheaves are the canonical zig-zag modules, adapted to the stratification by critical values.

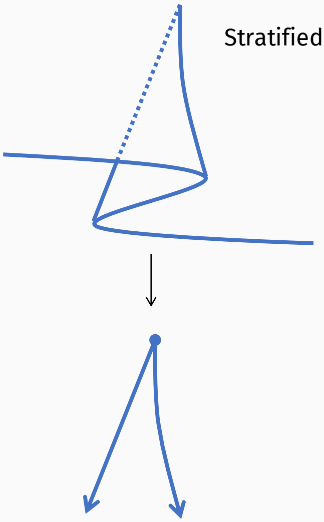


Level Set Barcodes have interesting endpoints!

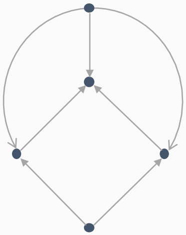
Related by Duality

Whitney's Cusp Revisited

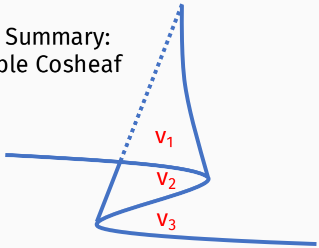
Stratified Map



Associated Entrance Path Category

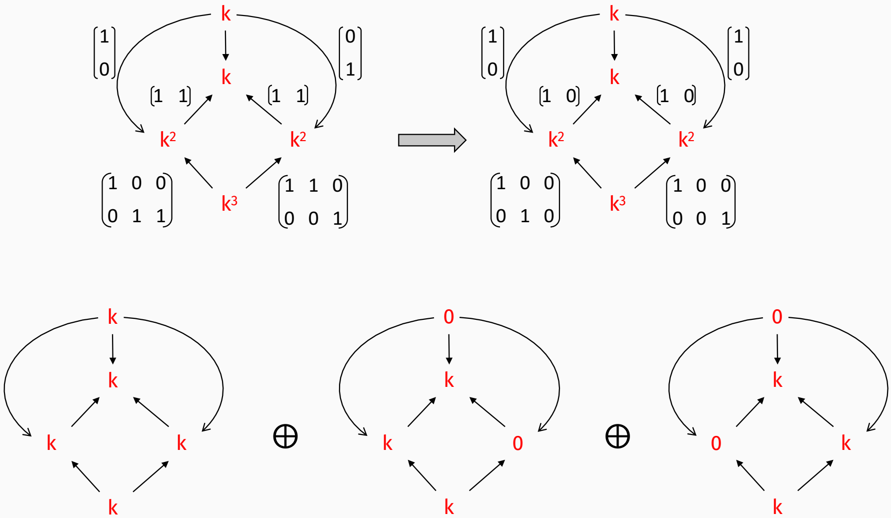


Algebraic Summary: Constructible Cosheaf

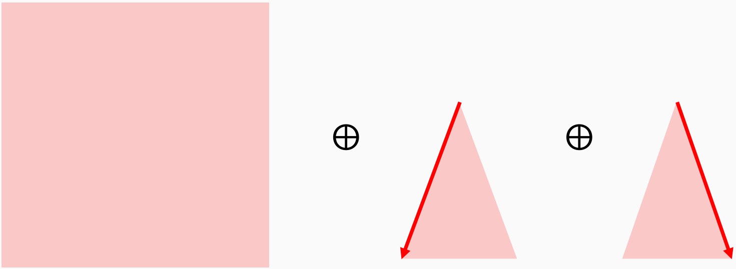


$$\begin{tikzcd}
 & k \arrow[d] \arrow[dl, bend right, "\begin{bmatrix}1\\0\end{bmatrix}"'] \arrow[dr, bend left, "\begin{bmatrix}0\\1\end{bmatrix}"] & \\
 & k & \\
k^2 \arrow[ur, "\begin{bmatrix}1 & 1\end{bmatrix}"] & & k^2 \arrow[ul, "\begin{bmatrix}1 & 1\end{bmatrix}"'] \\
 & k^3 \arrow[ul, "\begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\end{pmatrix}"] \arrow[ur, "\begin{pmatrix}1 & 1 & 0\\0 & 0 & 1\end{pmatrix}"'] &
\end{tikzcd}$$

“Unbreakable Summands” = Indecomposables



Higher-Dimensional Barcodes?



PROBLEM: Not every constructible cosheaf decomposes "nicely".

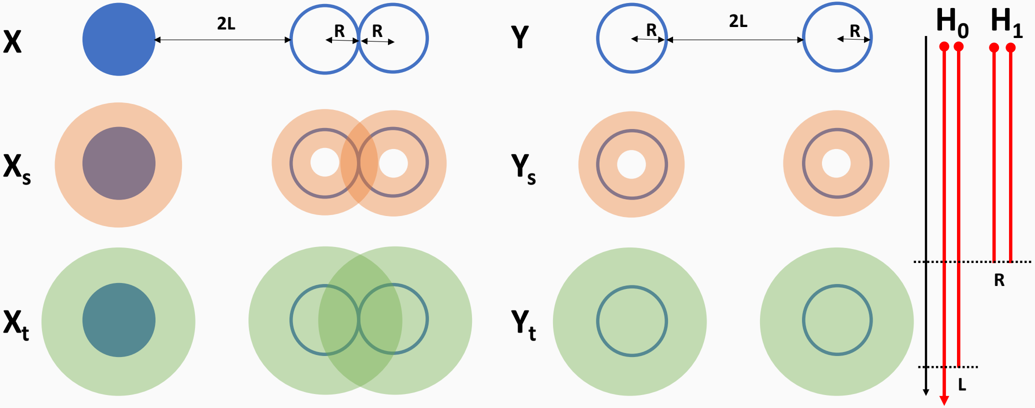
SOLUTION: Abandon All Indecomposables, Ye Who Enter Here

One should look at "stable" invariants or generalized rank invariants. Consider their Möbius Inversions à la MacPherson and Patel; Gulen and McCleary.

Decorated Merge Trees

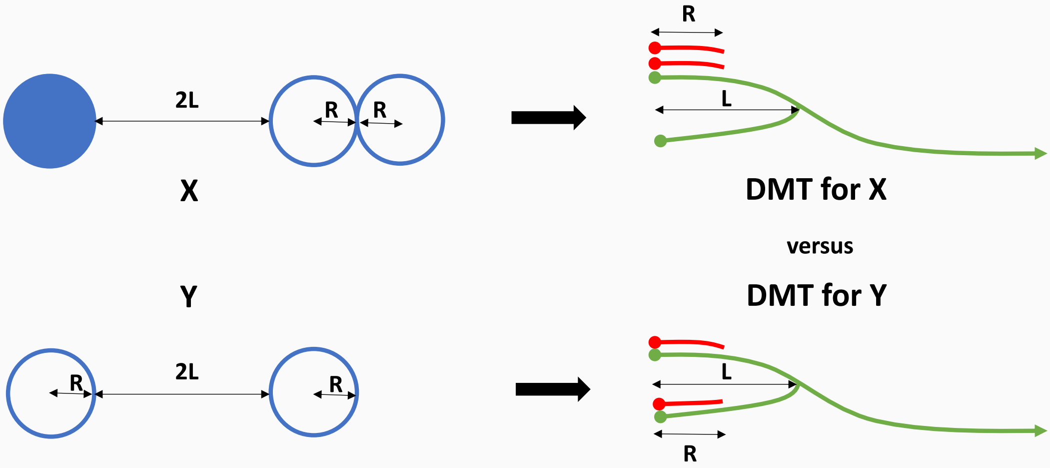
The Challenge

We want to distinguish the offset filtrations of X and Y using a minimal data structure.



The Cartoon Solution

A decorated merge tree should distinguishes these.

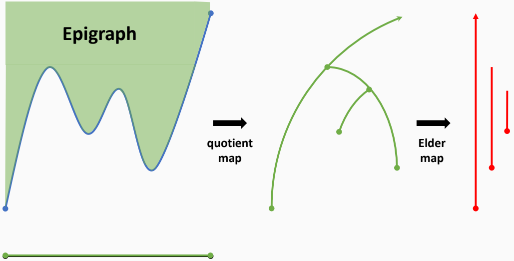


The Sheaf-Theoretic Solution

Sublevel set persistent homology is just the study of the Leray sheaves of the map π_f.

(Concrete) Decorated Merge Trees are defined by the Leray sheaves of the map q.

$$
\begin{tikzcd}
 & \mathcal{M}_f = E_f/\sim \arrow[dr, "\tilde{\pi}_f"] & \\
E_f \arrow[ur, "q"] \arrow[rr, "\pi_f"'] & & \mathbb{R}
\end{tikzcd}
$$



Problem

Different f have different $\mathcal{M}_f$. It is hard to compare sheaves on different base spaces.

Outline: Three Approaches to DMTs

We can define “enriched” topological summaries by changing the data category **D**.

Replace cosheaves of vector spaces with cosheaves of parameterized vector spaces.

This makes certain theorems easier to prove, namely, functional stability.

(Concrete DMT) $\mathcal{F}_n : \mathcal{M}_f \to \mathbf{Vect} \quad \Leftrightarrow \quad \widetilde{F}_n : \mathbb{R} \to \mathbf{pVect}$ **(Categorical DMT)**

For computation, restrict the DMT to assign a barcode to each leaf node.

(Barcode DMT) $\mathcal{B}\mathcal{F}_n : \mathcal{M}_f \to \mathbf{Barcodes}$

Tom Needham, along with Haibin Hang, Washington Mio, Osman Okutan and myself, implemented this last version of the DMT in Python:
`https://github.com/trneedham/Decorated-Merge-Trees`

Point Cloud Example

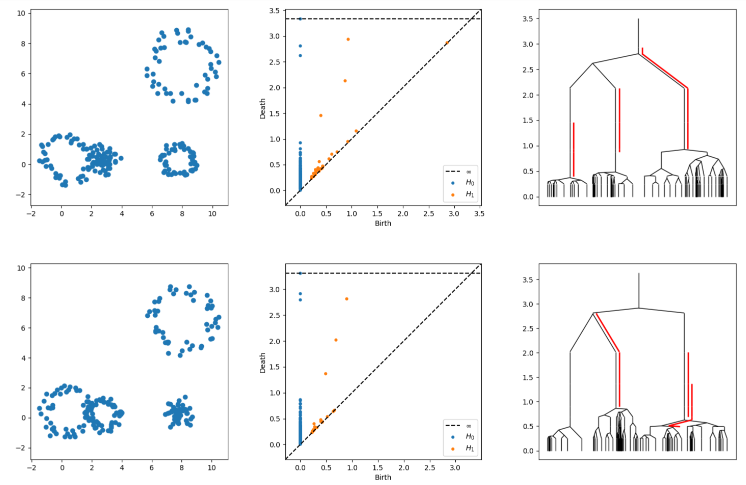


Figure Credit: Tom Needham (FSU)

Cycle Localization

Our DMT algorithm provides cycle localization *for free!*

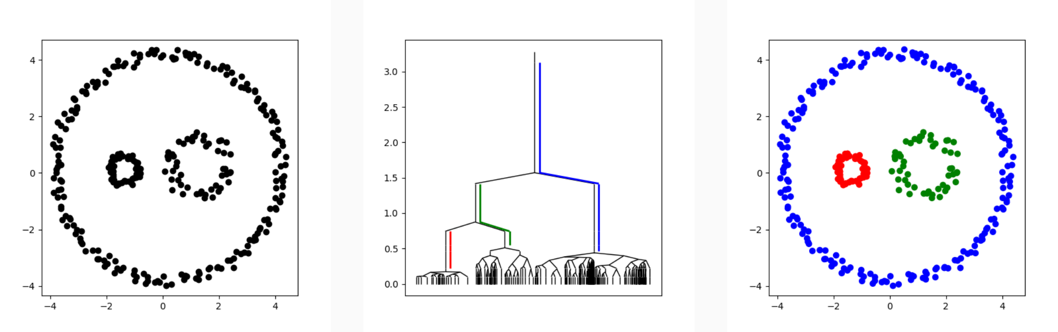


Figure Credit: Tom Needham (FSU)

Cycle Localization in Image Data

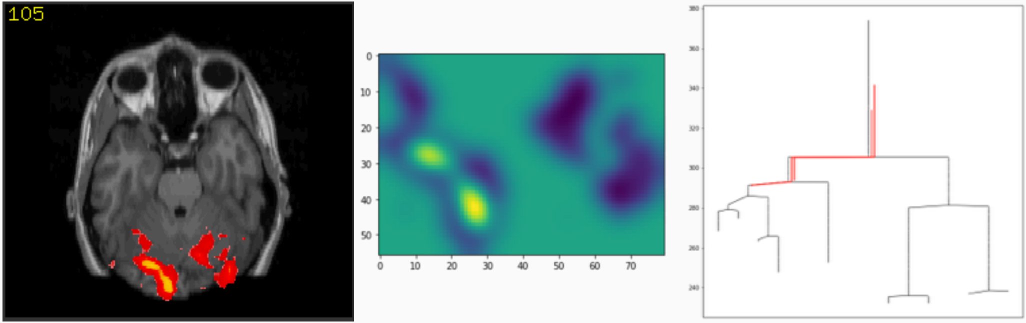


Figure Credit: Tom Needham (FSU)

Discrimination for Takens Embedding

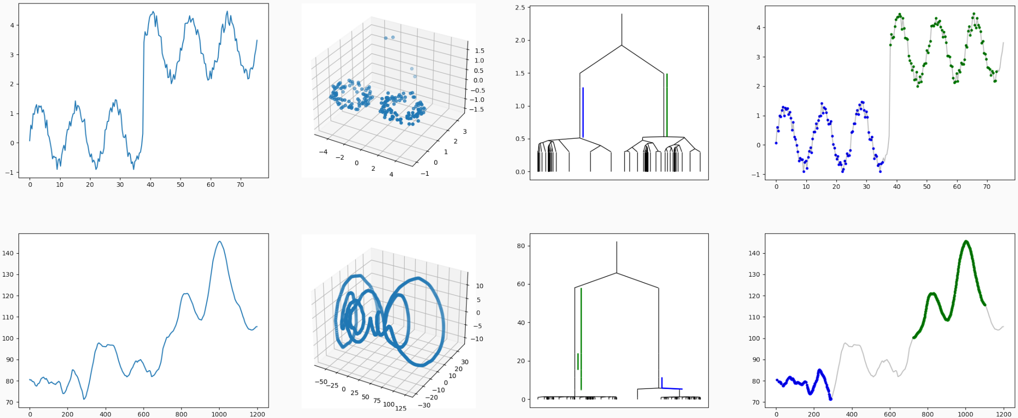


Figure Credit: Tom Needham (FSU)

Parameterizing by Components

If $f : X \to Y$ is a map of (locally connected) spaces, then we can express it as

$$f = \sqcup f_i : \coprod_{i \in \pi_0(X)} X_i \to \coprod_{j \in \pi_0(Y)} Y_j$$

This then induces a map

$$\oplus f_i : \bigoplus_i H_n(X_i) \to \bigoplus_j H_n(Y_j)$$

Let's leverage this into a categorical observation.

The Category of Parameterized Objects (Gabe Bainbridge, OSU PhD)

Definition

An **I-parameterized** object is a functor $I : S \to \mathbf{C}$, where S is any set.

Definition

A morphism from $I : S \to \mathbf{C}$ to $J : T \to \mathbf{C}$ consists of a map of sets $m : S \to T$ and a natural transformation $\alpha : I \Rightarrow J \circ m =: m^* J$.

Key Definition

Denote by **pC** the **category of parameterized objects** in $\mathbf{C}$.

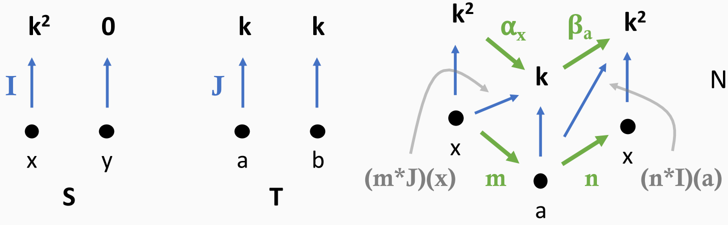
Non-Isomorphism in our Motivating Example

Definition

$I : S \to \mathbf{C}$ and $J : T \to \mathbf{C}$ are **isomorphic** if there are set maps $m : S \to T$ and $n : T \to S$ and transformations $\alpha : I \Rightarrow m^*J$ and $\beta : J \Rightarrow n^*I$ satisfying

$m^*\beta \circ \alpha = \mathrm{id}_I$ and $n^*\alpha \circ \beta = \mathrm{id}_J$. In particular, $n \circ m = id_S$ and $m \circ n = \mathrm{id}_T$.

Our motivating example reduces to the consideration of these two objects in **pVect**:



NOT ISOMORPHIC!

Natural Projection Functors

Note that if $\mathbf{C}$ has coproducts, then $\mathbf{pC}$ participates in the following diagram of categories and functors, where dom takes a parameterized object to its underlying parameterizing set and U takes each parameterized object to its coproduct.

$$\begin{tikzcd} & \mathbf{pC} \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "U"] & \\ \mathbf{Set} & & \mathbf{C} \end{tikzcd}$$

Lemma: Persistently Parameterized Space

Any persistent space of locally connected spaces $F : (\mathbb{R}, \leq) \to \mathbf{Top}^{\mathbf{lc}}$ has an associated **persistently parameterized space** $\tilde{F}$ where F is naturally isomorphic to the composition of functor $U \circ \tilde{F}$

$$
\begin{tikzcd}
 & (\mathbb{R}, \leq) \arrow[dl, "\pi_0 \circ \tilde{F}"'] \arrow[d, "\tilde{F}"] \arrow[dr, "F"] & \\
 & \mathbf{pTop^c} \arrow[dl, "\mathrm{dom}"] \arrow[dr, "U"'] & \\
\mathbf{Set} & & \mathbf{Top^{lc}} \arrow[ll, "\pi_0"]
\end{tikzcd}
$$

Categorical Definition of Decorated Merge Trees

Composition of $\tilde{F}$ with the homology functor $H_n : \mathbf{pTop} \to \mathbf{pVect}$ yields the **categorical decorated merge tree in degree** n, written $\tilde{F}_n$.

$$\begin{tikzcd}
 & (\mathbb{R}, \leq) \arrow[dl, "\pi_0 \circ \tilde{F}"'] \arrow[d, "\tilde{F}_n"] \arrow[dr, "H_n \circ F"] & \\
 & \mathbf{pVect} \arrow[dl, "\mathrm{dom}"] \arrow[dr, "U"'] & \\
\mathbf{Set} & & \mathbf{Vect}
\end{tikzcd}$$

Remarks on the Categorical Definition

Pros:

- For our motivating example, we identified an enrichment so that

$$\tilde{F}_n \ncong \tilde{G}_n \qquad \text{even though} \qquad H_n \circ F \cong H_n \circ G.$$

- Interleavings defined for $\mathbf{Fun}(\mathbb{R}, \mathbf{D})$. Applying $H : \mathbf{D} \to \mathbf{D}'$ is a Lipschitz map.

Easy Stability Theorems

MT Interleaving Distance $\leq$ DMT Interleaving Distance $\geq$ Bottleneck Distance

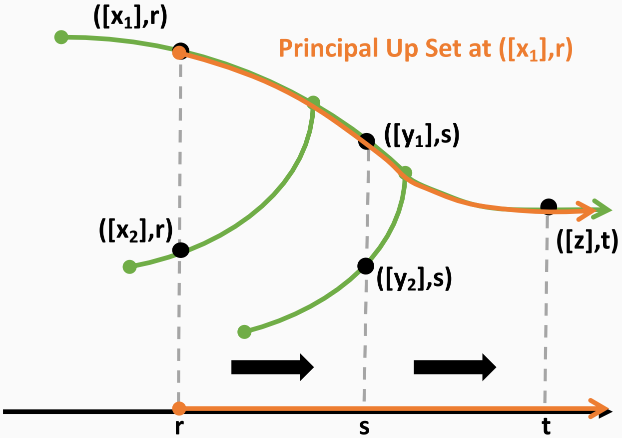
Cons:

- Ease of theorems comes at the expense of abstraction.
- Not immediately obvious how barcodes actually "sit on top of" the merge tree.

Restricting the Concrete DMT

Recall that $\mathcal{F} : \mathcal{M}_f \to \mathbf{Vect}$ is the Leray (co)sheaf on the merge tree.

We can restrict this to a totally ordered subset and extract a barcode there.



The Barcode DMT

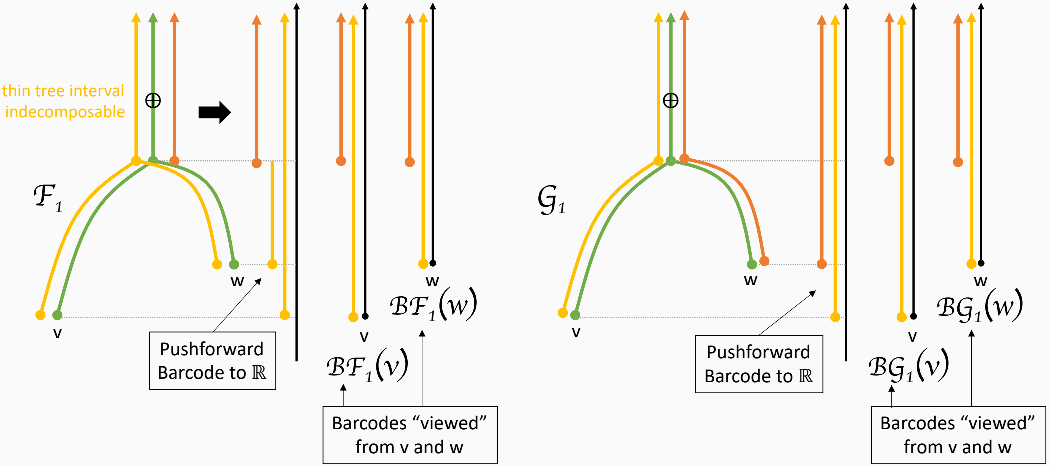
Barcode Decorated Merge Tree

Given $F : \mathbb{R} \to \textbf{Top}^{\textbf{lc}}$, the **barcode DMT** assigns to each $p = ([x], r) \in \mathcal{M}_F$ the barcode of the restricted Leray cosheaf, i.e. $\mathcal{F}_n|_{U_p}$. This defines

$$\mathcal{B}\mathcal{F}_n : \mathcal{M}_F \to \textbf{Barcodes}.$$

Note that whenever $p = ([x], r) \preccurlyeq ([y], s) = q$ we have $\mathcal{B}_F(q) = \mathcal{B}_F(p)|_{[s,\infty)}$, so it suffices to specify one barcode for each leaf node.

The Barcode Transform is Not Injective



A Hierarchy of Distances

By considering interleavings of merge trees that also infimize the bottleneck distance between the pullback of these restricted (co)sheaves, we obtain a new **decorated bottleneck distance**.

This fits into the following hierarchy:

Theorem (C. + Hang, Mio, Needham, Okutan)

For $\mathbb{R}$-spaces $X_f := f : X \to \mathbb{R}$ and $Y_g := g : Y \to \mathbb{R}$ we have the following sequence of bounds

$$d_{MT}(\mathcal{M}_f, \mathcal{M}_g) \leq d_{DB}(\mathcal{BF}_n, \mathcal{BG}_n) \leq d_{DMT}(\tilde{F}_n, \tilde{G}_n) \leq \delta_I(X_f, Y_g)$$

There is no relationship between the bottleneck and decorated bottleneck distance!

Some Comments

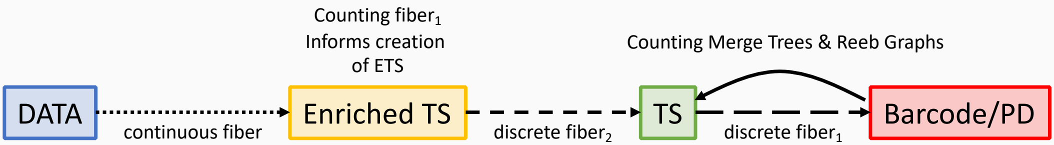
- There is an obvious generalization to Reeb graphs.
- Extracting tractable invariants from Decorated Reeb Graphs is a challenge.

Sheaf Theory in The Large

Bird's Eye Perspective

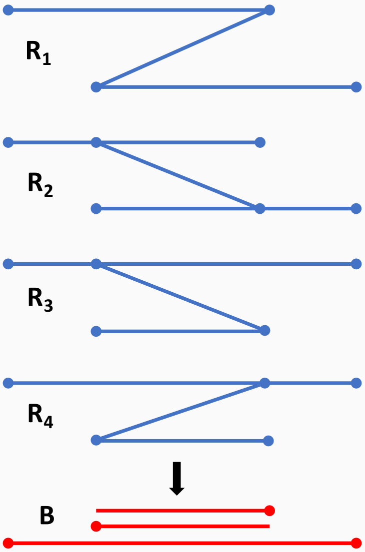
We have considered the TDA pipeline by studying individual inputs and outputs.

Most of these steps can be summarized using sheaf theory.



We now recurse and study the TDA pipeline as a map, algebratizing it accordingly.

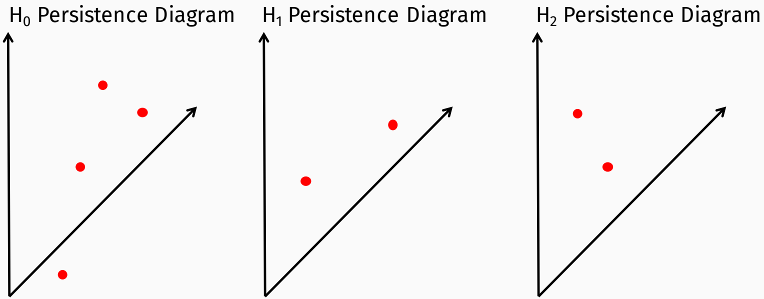
Open Problem!



The Persistence Map

The persistence pipeline is actually a sequence of spaces and maps:

$$f : X \to \mathbb{R} \qquad \rightsquigarrow \qquad F : t \mapsto X_{\leq t} \mapsto H_n(X_{\leq t}) \qquad \rightsquigarrow \qquad \mathrm{PD}(f)_n$$



Questions

What is the image of this map? and **What are its fibers?**

Realization Problems for Persistence

The "Realization Problem" asks to provide a $f : X \to \mathbb{R}$ with a given barcode.

There are uncountably many realizations so we often pass to equivalence classes.

Let's bypass functions and work with the following step in the pipeline:

$$\text{Free} : \mathbf{Fun}(\mathbb{R}, \mathbf{Set}) \to \mathbf{Fun}(\mathbb{R}, \mathbf{Vect}) \quad \pi_0(F(s)) \to \pi_0(F(t)) \quad \rightsquigarrow \quad H_0(F(s)) \to H_0(F(t))$$

Both of these categories can be topologized using the interleaving distance.

The map taking a merge tree to its barcode is 1-Lipschitz.

From Trees to Barcodes

The Elder Rule

There is an equivalent way of describing the Free functor just described.

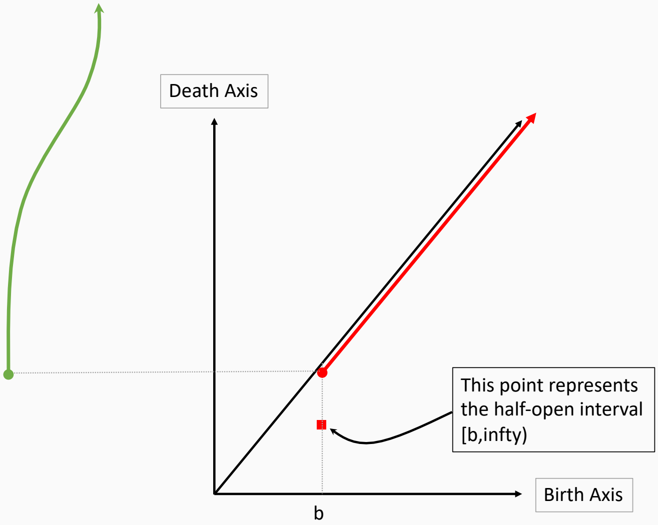
The Elder Rule (apocryphal, but proved in C. '17)

Given a merge tree $\pi : T \to \mathbb{R}$:

1. Start sweeping from $-\infty$ to $+\infty$
2. Whenever you encounter a leaf node of T, start drawing an interval, aka "bar" in a barcode.
3. Bars correspond to branches until two branches meet.
4. The bar coming from the "older" branch continues, the younger one dies.

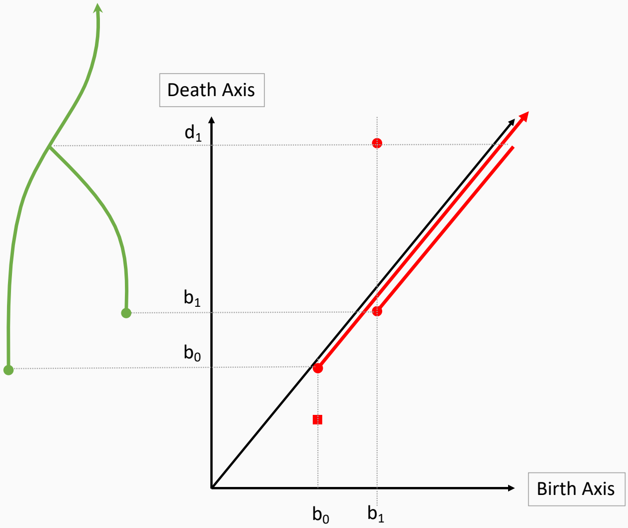
The Realization Problem for Merge Trees

Merge Tree
that
“Realizes B”



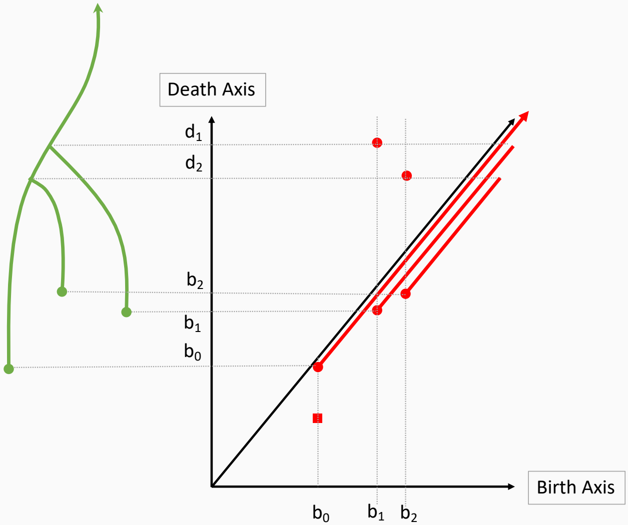
The Realization Problem for Merge Trees

Merge Tree
that
"Realizes B"



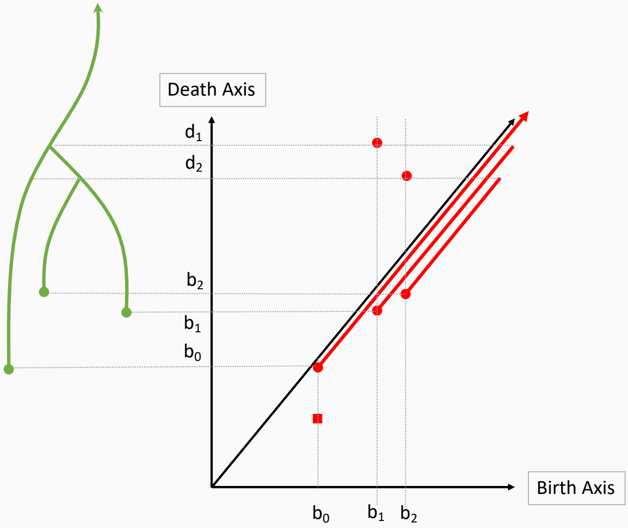
The Realization Problem for Merge Trees

2 Merge Trees
realize B.
Here is MT1



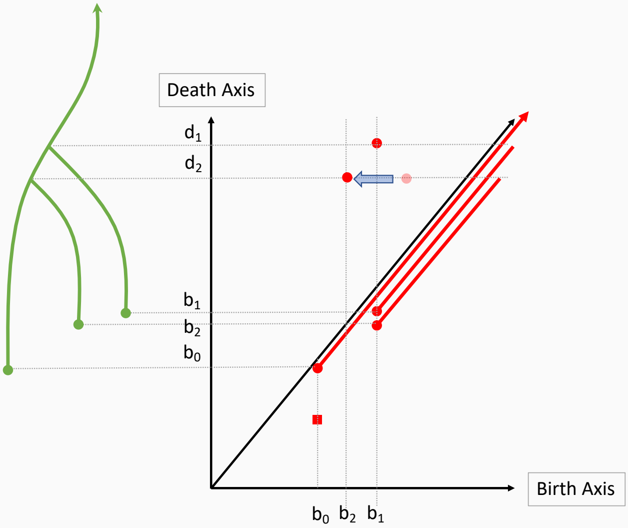
The Realization Problem for Merge Trees

2 Merge Trees
realize B.
Here is MT2



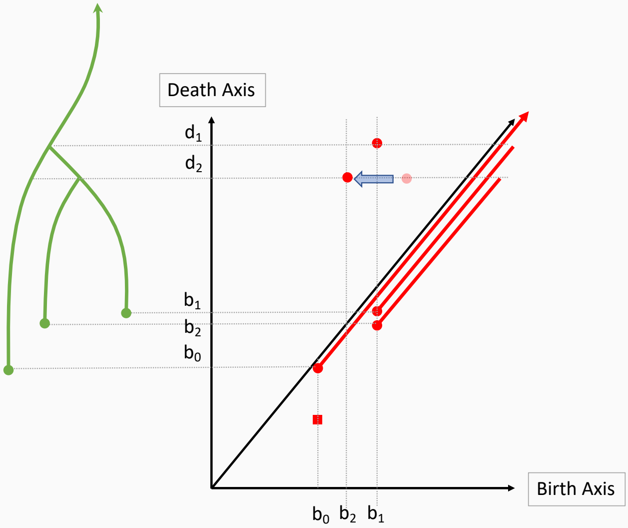
Change of Birth Order

1 Merge Tree
Realizes B



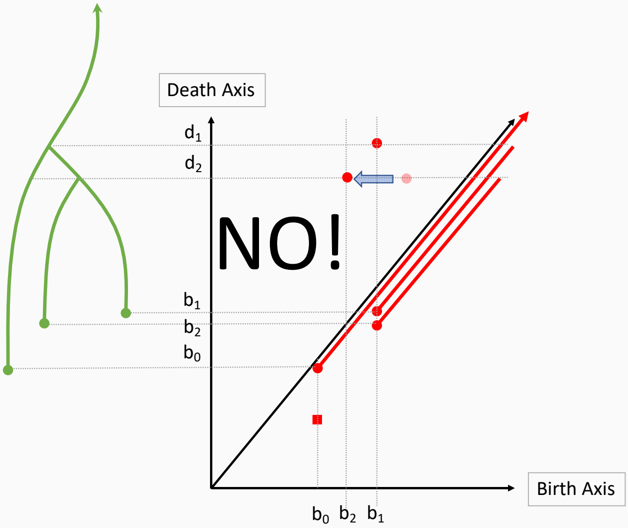
Possible Realization?

Is this a Realization of B?



Elder Rule

ELDER RULE SAYS...



Enumerating the Fiber of the Elder Map

Theorem (C. '17; Garin, Hess, Kanari '20)

Fix a barcode B where every endpoint is distinct:

$$B = \{[b_0, \infty); [b_1, d_1); \ldots ; [b_n, d_n)\}$$

Further, we assume $I_0 = [b_0, \infty)$ and $I_j := [b_j, d_j)$ satisfy

- (**Containment**) $I_j \subset I_0$ for all $j \geq 1$ and
- (**Increasing Birth Times**) $b_1 < b_2 < \cdots < b_n$.

Set $\mu_B(I_j) = \#\{k < j \mid d_j < d_k\}$. The number of merge trees realizing B is

$$TRN(B) := R(B) := \prod_{j=1}^{n} \mu_B(I_j)$$

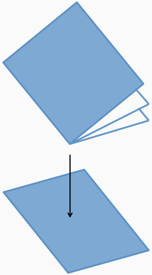
Significance: Stratified Covering Spaces

Containment Poset

The barcode $B = \{I_j\}$ forms a poset, ordered by containment of intervals.

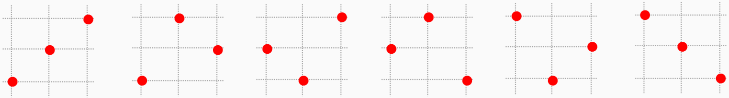
The space of persistence diagrams/barcodes is *stratified* by the containment poset.

The Elder Map defines a *stratified covering space* or a **Set**-valued constructible cosheaf over barcode space. The number of sheets over each top-dimensional stratum is $R(B)$.



Inversion Vectors

Adélie Garin, Kathryn Hess, and Lida Kanari observed that *generic* persistence diagrams can be viewed as permutations, i.e. elements of the symmetric group.



permutation	id	(23)	(12)	(123)	(132)	(13)
left inversion vector	(0,0,0)	(0,0,1)	(0,1,0)	(0,0,2)	(0,1,1)	(0,1,2)
number of inversions	0	1	1	2	2	3

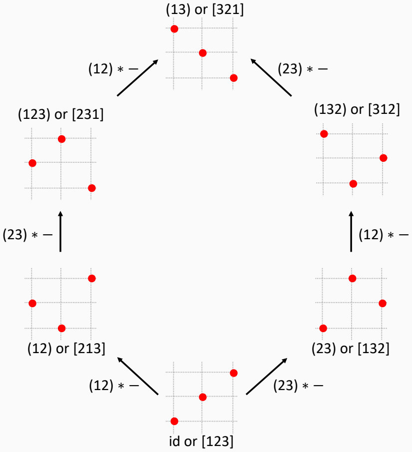
Observation by GHK + C. + Brendan Mallery and Jordan DeSha

Let σ be the permutation type of a generic B and let $\ell(\sigma)$ denote the left-inversion vector of σ, then $R(B) = \prod_{i=1}^{n}(\ell_i(B) + 1)$

Tree Realization Number is Bruhat Order Preserving

Lemma (CDGHKM '21)

If $\sigma, \sigma' \in S_n$ are such that $\sigma < \sigma'$ in the (left) Bruhat order, then $R(\sigma) < R(\sigma')$.



TRN on the Cayley Graph

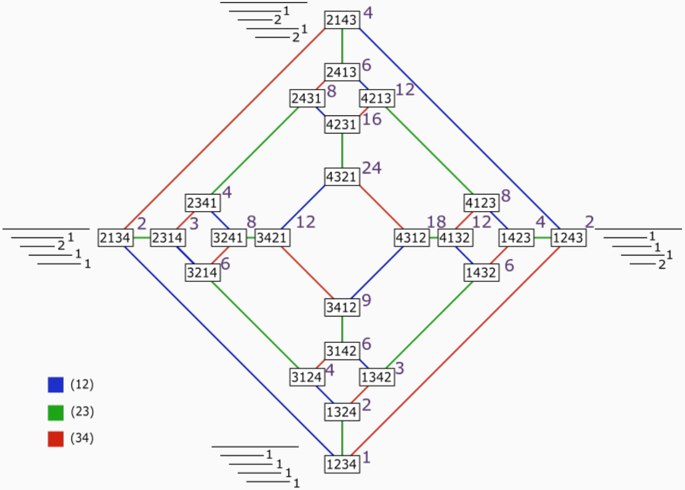


Figure Credit: Adélie Garin (EPFL)

The Average TRN

Theorem (CDGHKM '21)

If $\mathbb{U}$ denotes the uniform distribution on the symmetric group, then

$$\mathbb{E}_{\mathbb{U}}(R(B)) = \frac{(n+1)!}{2^n}.$$

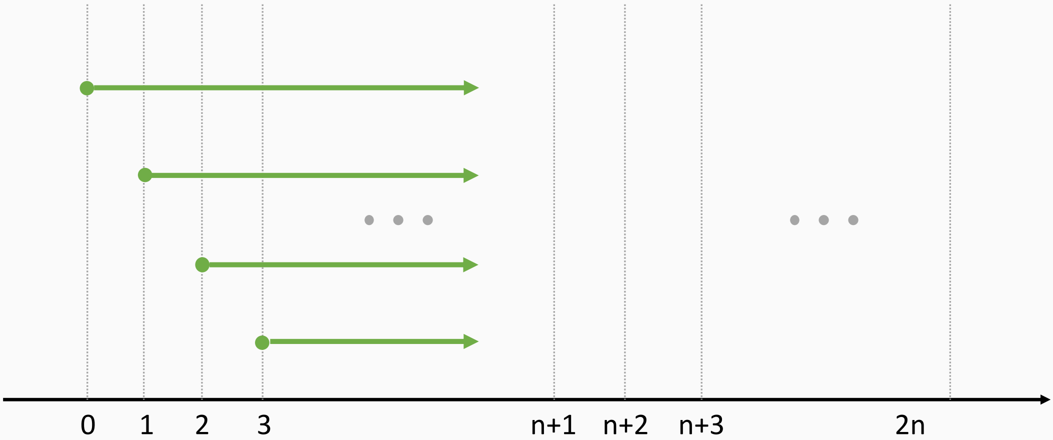
Counting Combinatorial Merge Trees

1. Top $2n$-dimensional strata of barcode space $\mapsto S_n$, so there are $n!$ many.
2. Merge trees form a (stratified) covering space of barcode space, so the sum

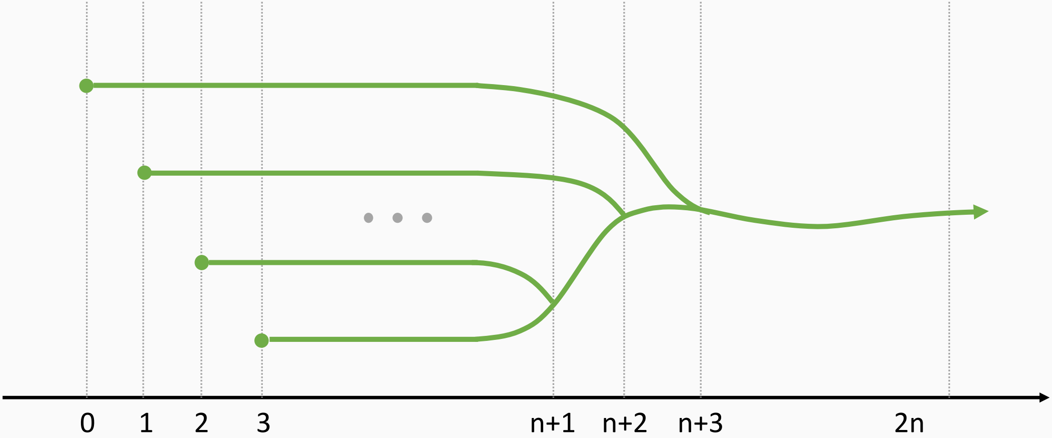
$$\sum_{\sigma \in S_n} R(B_\sigma)$$

   counts top $2n$-dimensional strata.
3. Both the strata of barcode space and merge tree space are **convex.**

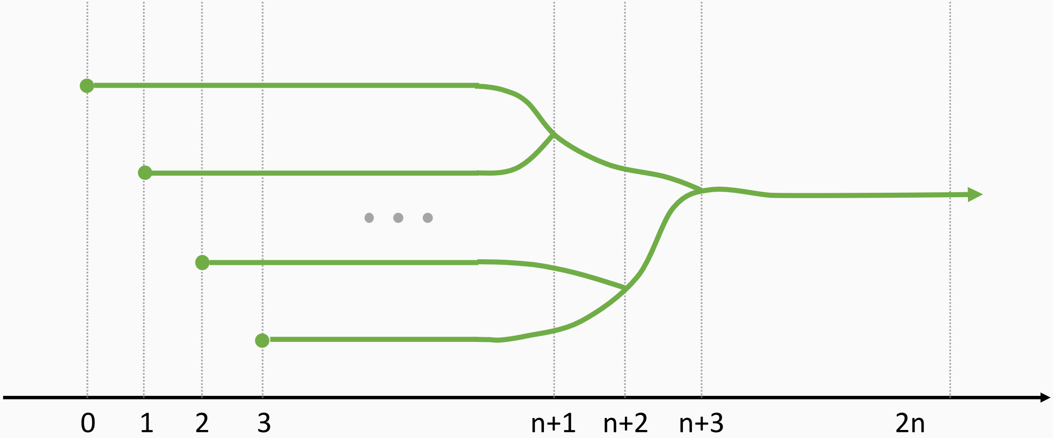
Counting Standard Merge Trees



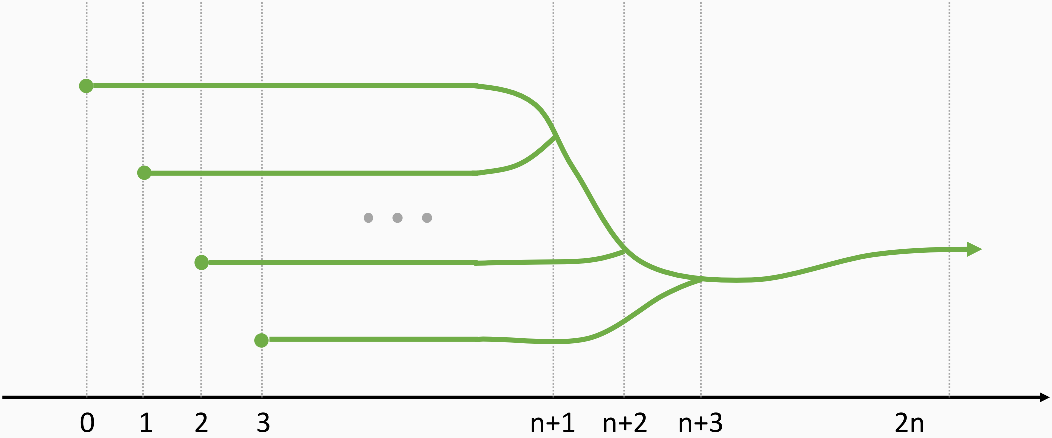
Counting Standard Merge Trees



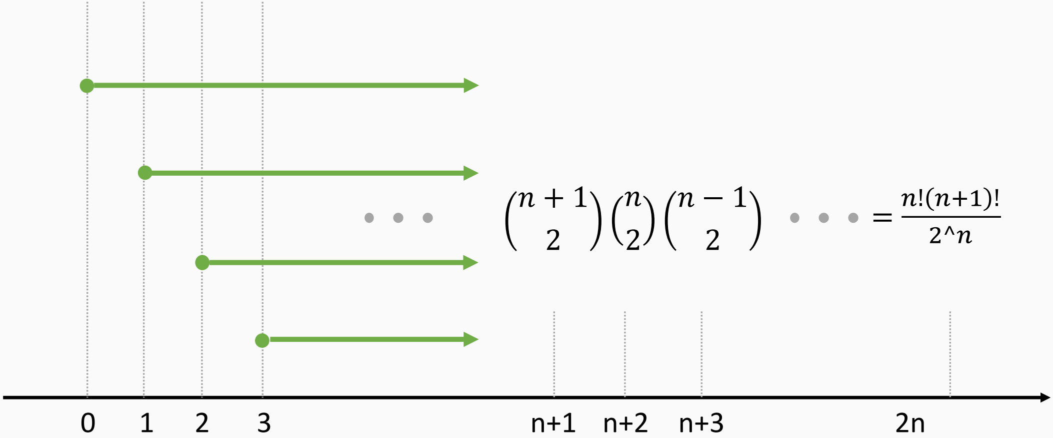
Counting Standard Merge Trees



Counting Standard Merge Trees



Counting Standard Merge Trees



$$\binom{n+1}{2}\binom{n}{2}\binom{n-1}{2}\cdots = \frac{n!(n+1)!}{2\hat{}n}$$

Maximal Chains in the Lattice of Partitions

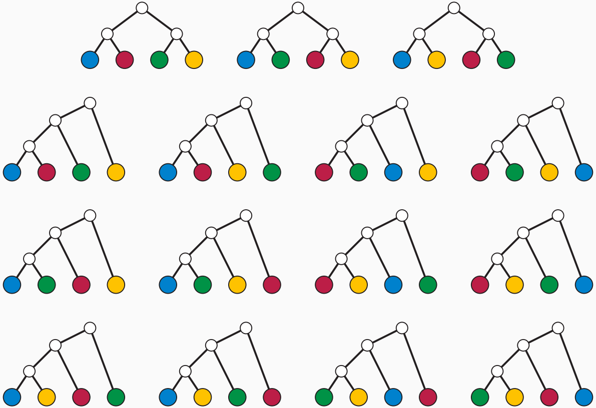
Theorem (CDGHKM '21)

The Elder Map stratifies MT Space so that top-dimensional strata are in bijection with maximal chains in the lattice of partitions. In summary,

$$\sum_{\sigma \in S_n} R(B_\sigma) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} (l_i(\sigma) + 1) = \frac{(n+1)!n!}{2^n}.$$

cf. BHV space, where orthants are counted by $(2n-1)!!$, where $n = \#\text{leaves} - 1$

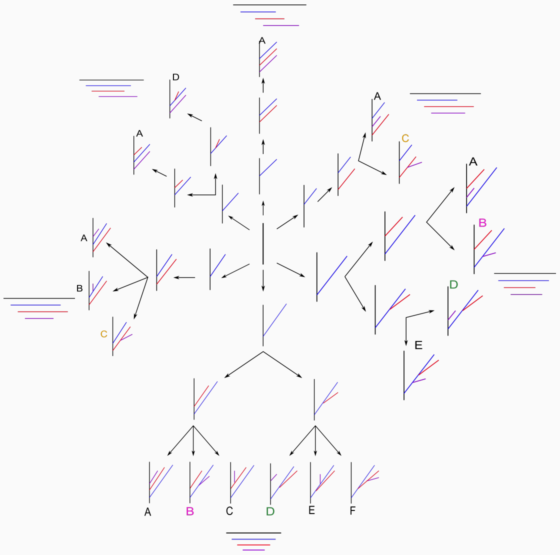
BHV Trees



For 4 leaves, there are 15 BHV topologies.

Figure Credit: Wikipedia

18 MT Trees (Figure Credit: Adélie Garin)



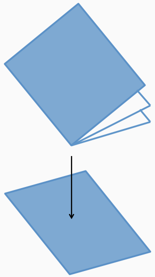
Remarks on MT Space versus BHV Space

Merge Tree space is very different from BHV space.

- Points in MT Space correspond to *isomorphism* classes of merge trees.
- Generically we can label leaf nodes by birth time, but this is not continuous.
- Moreover, a given orthant of BHV space (split topology type) does not have a uniquely associated permutation type of barcode, but we can bound this difference precisely. See Prop 3.23 of `https://arxiv.org/abs/2107.11212`
- Understanding the stratified space structure is critical for doing good statistics.

An Aside

Constructible cosheaves of sets are equivalent to stratified covering spaces.

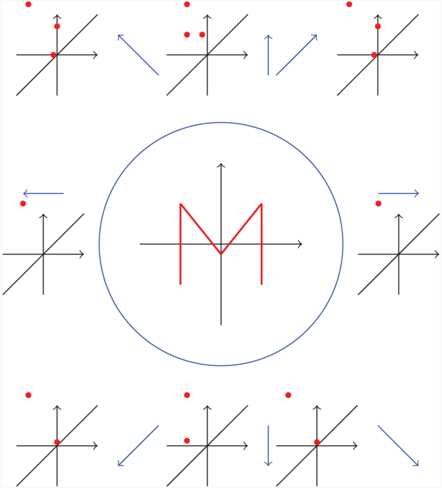


Morally, this allows us to lift metrics from barcode space to merge tree space.

How to lift the Wasserstein p-distance on barcodes to merge trees? See "Presentation Based Metrics on Merge Trees" by Tung Lam, R. Cardona, M. Lesnick and C. where we avoid cosheaves entirely: `https://youtu.be/b9x-Esq6nIE`

The Big PHT Sheaf

Persistent Homology Transform



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The Basic Idea (w/ Shreya Arya and Sayan Mukherjee)

Definition

Given $M \subseteq \mathbb{R}^d$ we have $Z_M := \{(x, v, t) \in M \times S^{d-1} \times \mathbb{R} \mid x \cdot v \leq t\}$.

The **derived PHT** of M is the right derived pushforward of the constant sheaf onto $S^{d-1} \times \mathbb{R}$.

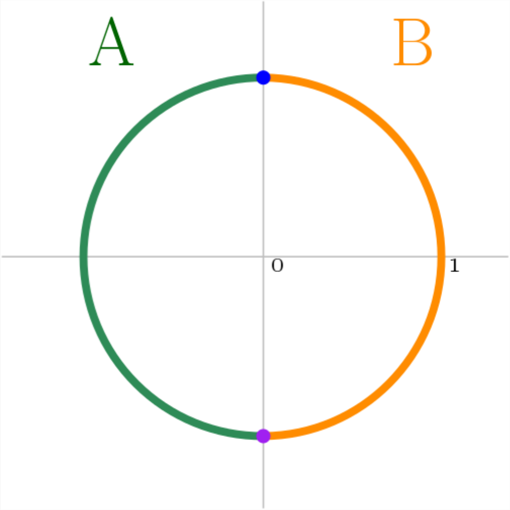
Notice that if we include a subset $A \hookrightarrow M$ this should induce a map of sheaves

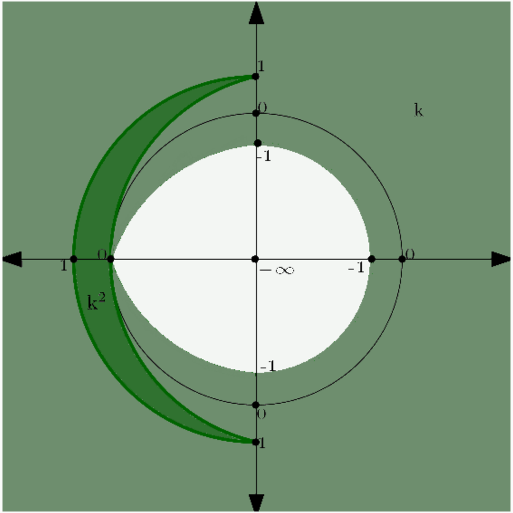
$$\mathrm{PHT}(M) \Rightarrow \mathrm{PHT}(A)$$

Theorem (ACM '21)

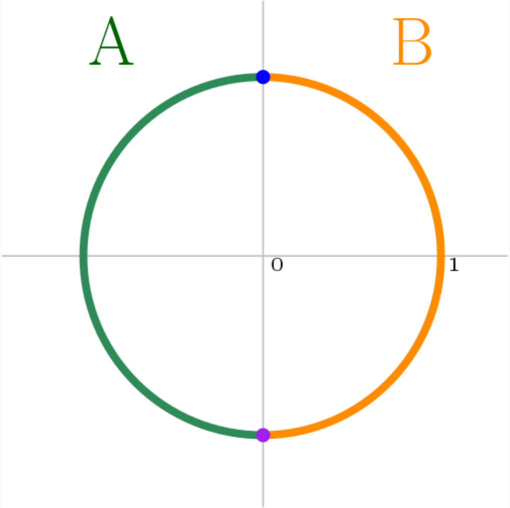
PHT is a (hyper) sheaf on the o-minimal site of constructible sets.

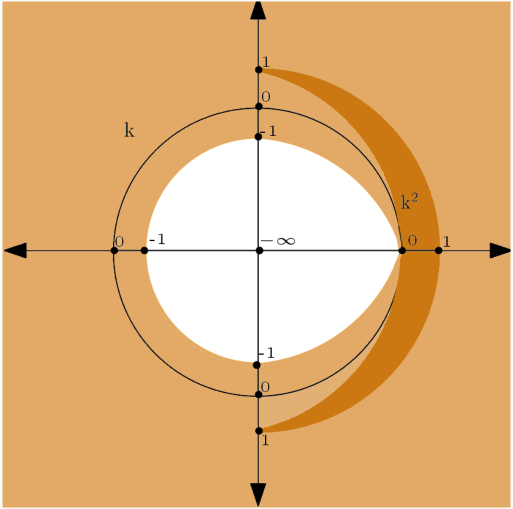
PHT Sheaf Example from Shreya Arya (Duke)



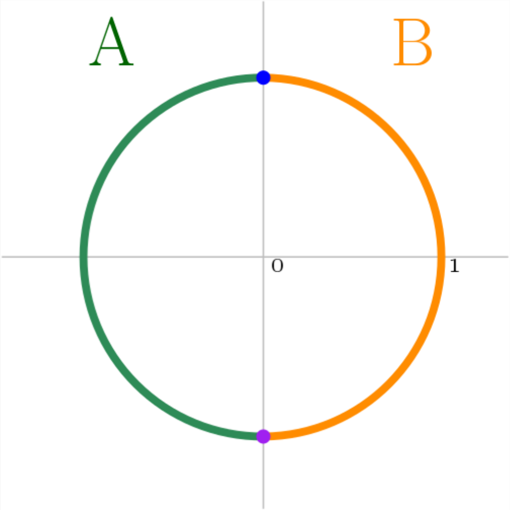


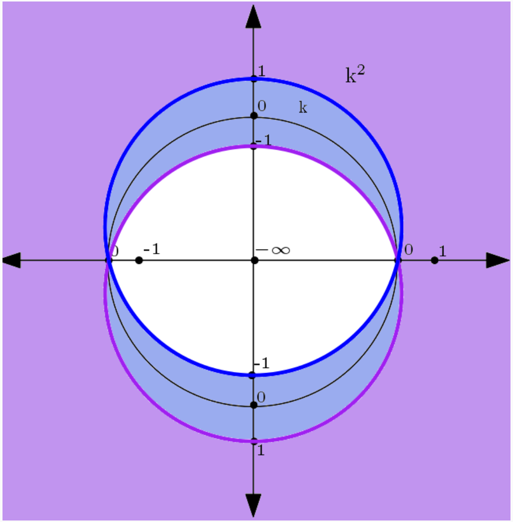
PHT Sheaf Example from Shreya Arya (Duke)



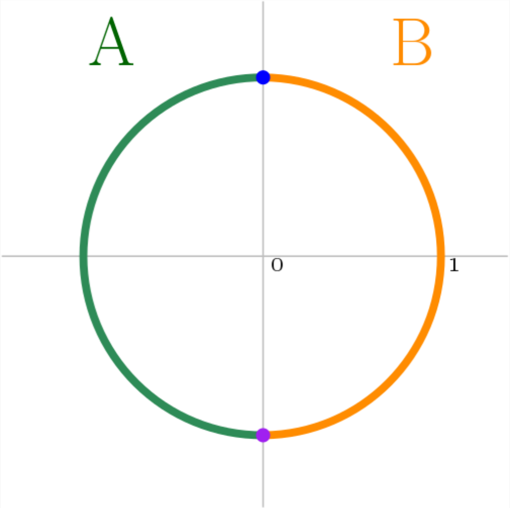


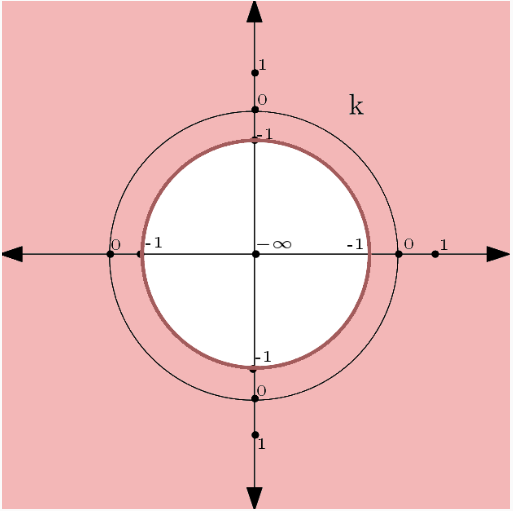
PHT Sheaf Example from Shreya Arya (Duke)





PHT Sheaf Example from Shreya Arya (Duke)





PHT in Degree 0 is Enough

- For a convex subset $A \subseteq \mathbb{R}^d$, all of the PHT is concentrated in degree 0.
- If we view a polyhedron $M \subseteq \mathbb{R}^d$ as glued together convex shapes, then we can recover $\mathrm{PHT}(M)$ completely in terms of $\mathrm{PHT}^0(M_i)$, where $\{M_i\}$ is a locally finite convex cover of X.

Theorem (Arya, C., Mukherjee '21)

For the simplicial complex $M \in \mathbb{R}^d$ and cover $\mathcal{V} = \{M_i\}_{i \in I}$ of M, $\mathrm{PHT}^n(M)$ is the n-th cohomology of the following complex of *sheaves*:

$$0 \to \oplus_{i \in I} \mathrm{PHT}^0(M_i) \to \oplus_{i<j} \mathrm{PHT}^0(M_i \cap M_j) \to \cdots$$

where the $\cdots$ represents the higher intersection terms.

A Convergence Result for the PHT

The PHT type distance is

$$d_I(\mathrm{PHT}(X), \mathrm{PHT}(Y)) = \int_{S^{d-1}} \delta_I(X_{f_v}, Y_{g_v}) dv.$$

Theorem (Arya, C., Mukherjee '21)

Let M be a compact submanifold of $\mathbb{R}^d$ with condition number τ. Let $\bar{x} = \{x_1, .., x_n\}$ be a set of n points drawn independently and identically from a uniform probability measure on M. Let $0 < \epsilon < \frac{\tau}{4 \text{ vol } (S^{d-1})}$. Let $U = \cup_{x \in \bar{x}} B_\epsilon(x)$ be the union of the open balls of radius ϵ around the sample points. Let K be the nerve of U. Then for all $n > \beta_1(\log \beta_2 + \log \frac{1}{\delta})$ we have that, with probability $> 1 - \delta$,

$$d_I(\mathrm{PHT}(M), \mathrm{PHT}(K)) \leq \epsilon$$

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Thank You!