# Decorated Merge Trees for Persistent Topology


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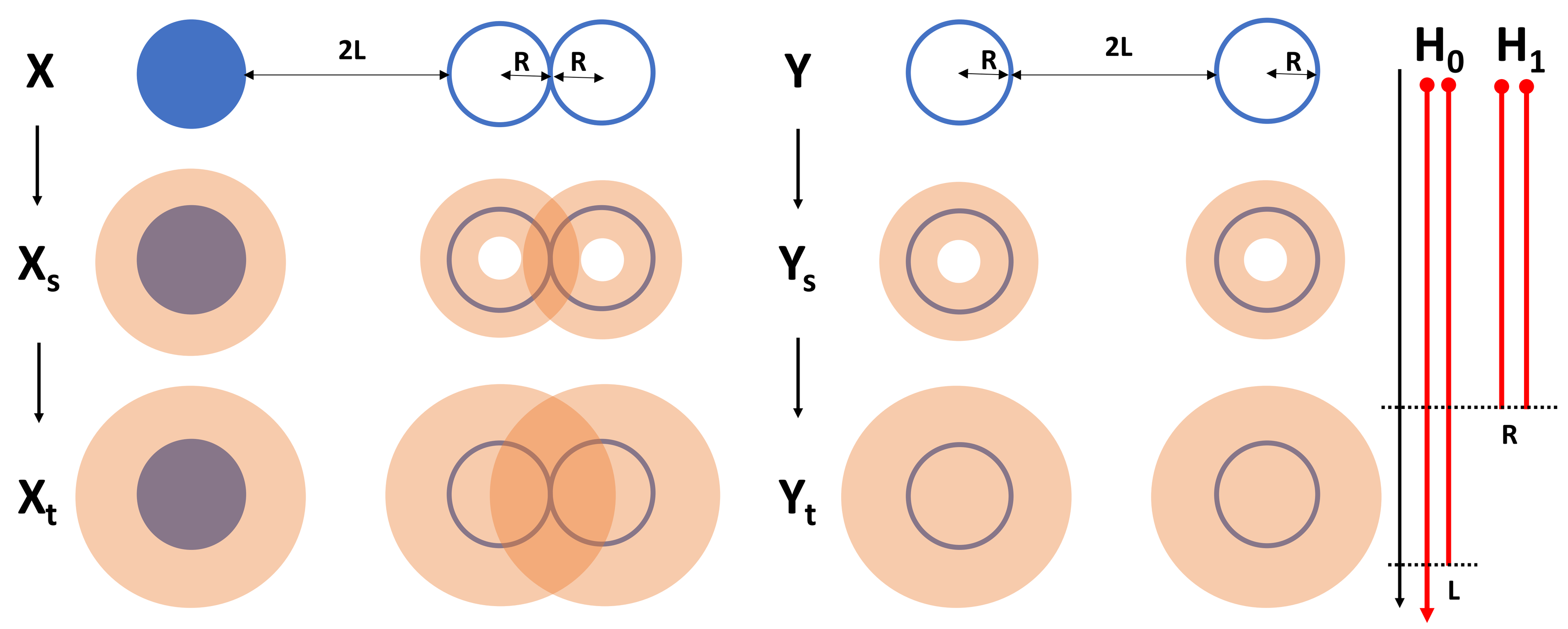
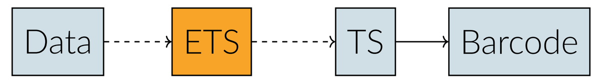
## The Motivating Example

How do we differentiate the subsets $X$ and $Y$ using persistent homology?



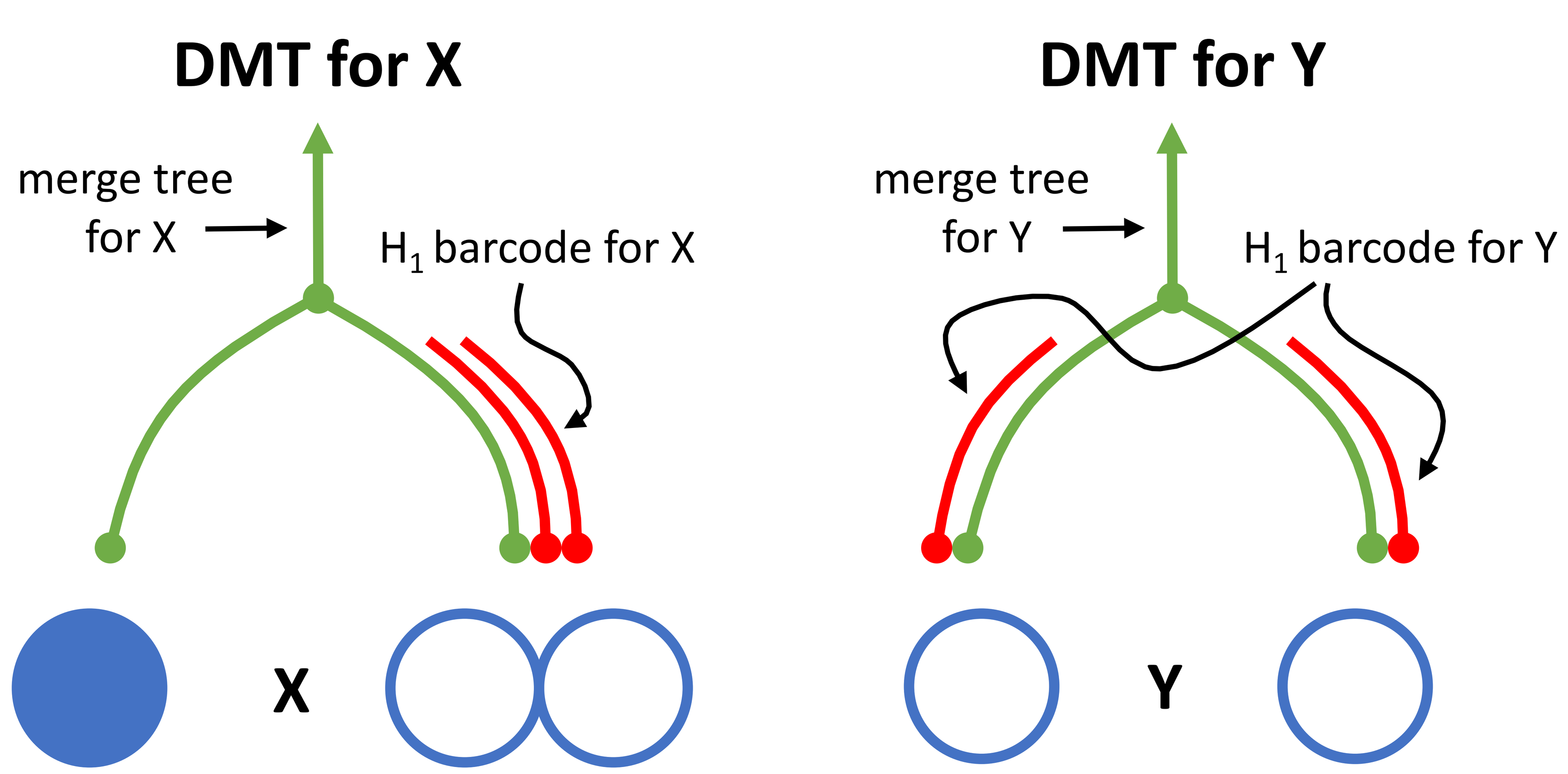

## Inverse Problems and Enriched Topological Summaries

The motivating example shows that persistent homology has an interesting inverse problem. The **decorated merge tree**, which knits together connected component ($\pi_0$) and homological information, is an example of a new **enriched topological summary (ETS)**, which fits into the existing pipeline of topological summaries




## Cartoon Definition of the DMT

Decorated Merge Trees enrich merge trees with persistent homology barcodes that allow us to distinguish the two subsets.




## DMTs Three Different Ways

DMTs can be approached three different ways, with each approach having particular advantages and disadvantages. However, Definitions 1 and 2 are *equivalent* in a certain sense.

1. **A Categorical DMT** is defined as a functor from $(\mathbb{R}, \leq)$ to $\mathbf{pVect}$. This latter category is the **category of parameterized vector spaces**, which can be challenging to understand, but it provides a precise sense in how DMTs refine merge trees and persistent homology. This also provides instant proofs of stability.
2. **Concrete DMTs** are defined as functors from $(\mathcal{M}_F, \preccurlyeq)$, the merge tree viewed as a poset, to $\mathbf{Vect}$. These are also called representations or **tree modules**.
3. **Barcode DMTs** take a tree module and restrict the representation to the up set at a point, thereby obtaining a barcode attached to that point. This is the most combinatorial, but also loses information.

## The Category of Parameterized Objects

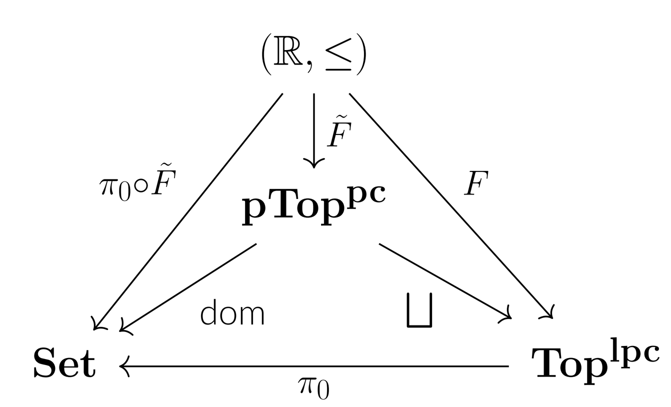
An **I-parameterized** object is a functor $I$ from a set $S \in \mathbf{Set}$, viewed as a discrete category, to a category $\mathbf{C}$, i.e. $I : S \to \mathbf{C}$.

A **morphism** from an $I$-parameterized object in $\mathbf{C}$ to a $J$-parameterized object, written $J : T \to \mathbf{C}$, consists of a map of sets $m : S \to T$ and a natural transformation $\alpha : I \Rightarrow J \circ m =: m^*J$.

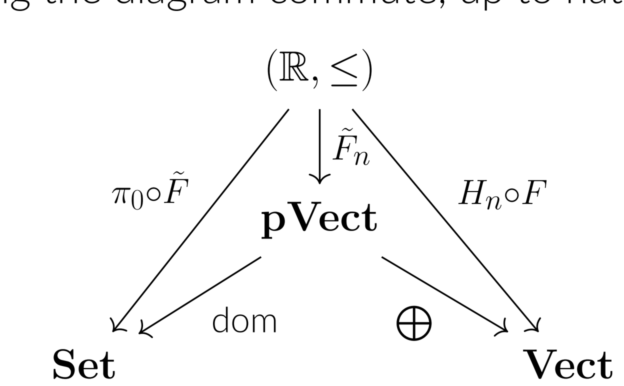
We denote by $\mathbf{pC}$ the **category of parameterized objects** in $\mathbf{C}$, whose objects are functors $I : \mathbf{I} \to \mathbf{C}$ for some set $\mathbf{I} \in \mathbf{Set}$ and whose morphisms are natural transformations $\alpha : I \Rightarrow J \circ m$.

## Refining Persistent Homology and Merge Trees

Any persistent space of locally path connected spaces $F : (\mathbb{R}, \leq) \to \mathbf{Top}^{\mathbf{lpc}}$ has an associated **persistently parameterized space** $\tilde{F}$ where $F$ is naturally isomorphic to the disjoint union of the spaces indexed over the merge tree.

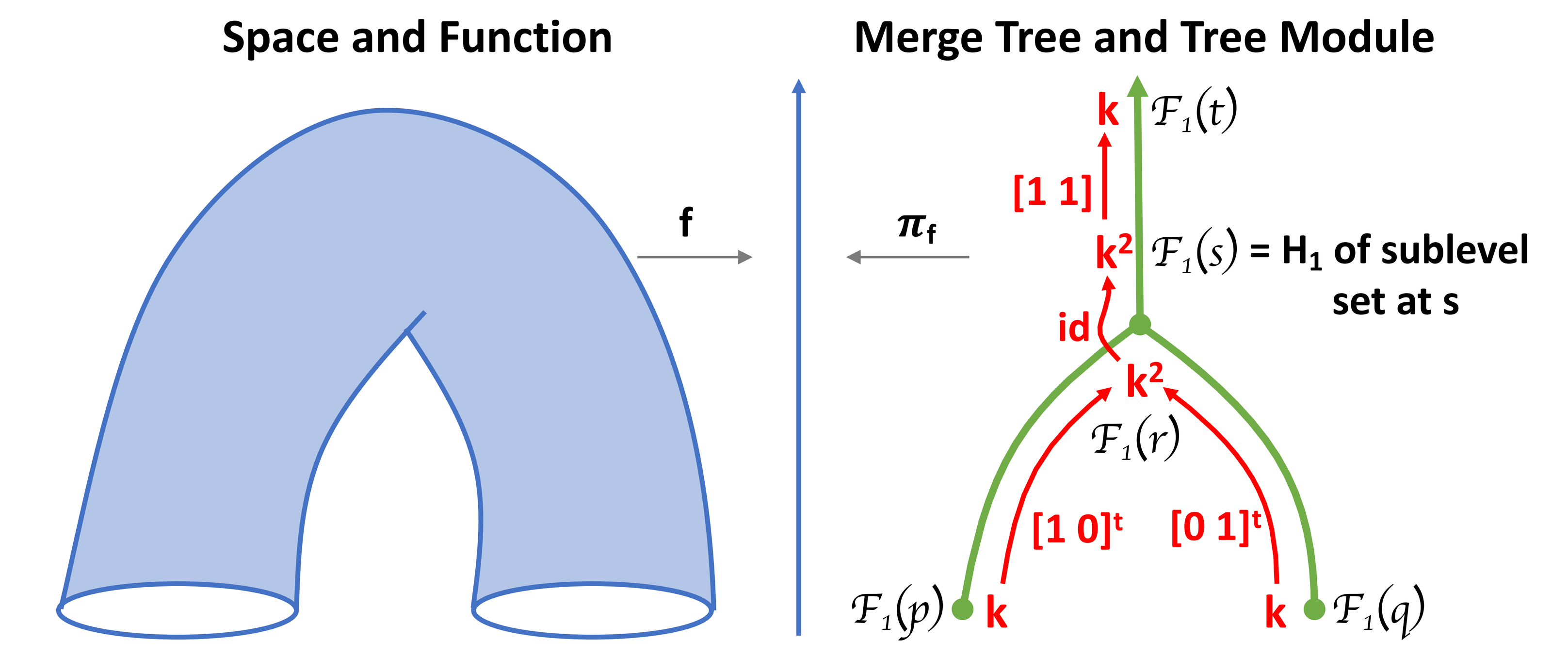



Composition of $\tilde{F}$ with the homology functor $H_n : \mathbf{pTop} \to \mathbf{pVect}$ yields the **categorical DMT in degree** $n$, written $\tilde{F}_n$, making the diagram commute, up to natural isomorphism.




Notice how we obtain merge trees and persistent homology in a single unified framework! Moreover the notion of interleaving of categorical DMTs comes for free.

## Tree Modules and Indecomposables

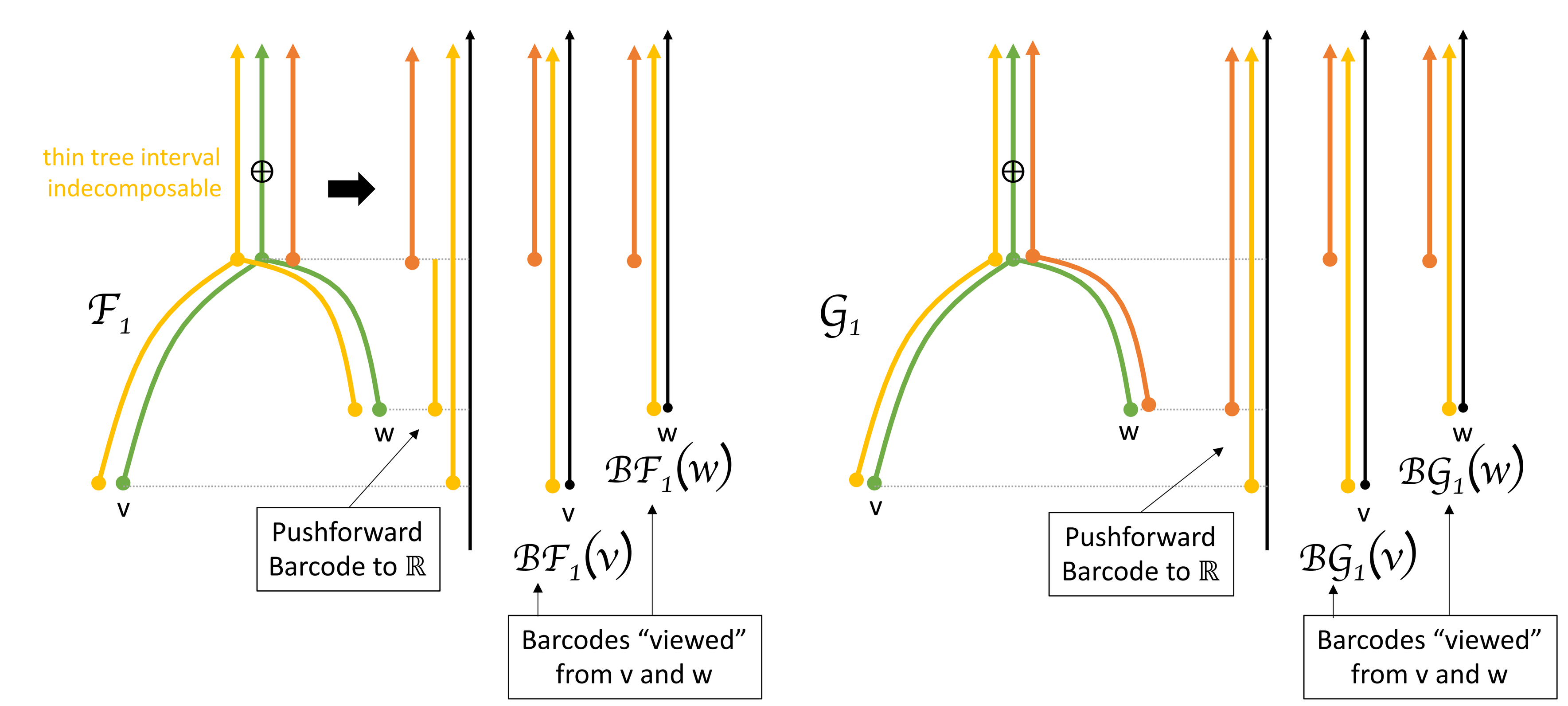



## Barcode DMTs

The assignment to each $p \in \mathcal{M}_F$ the barcode of the restricted persistent homology module $BC(\mathcal{F}|_{U_p})$ defines the **barcode DMT**:

$$\mathcal{BF} : \mathcal{M}_F \to \mathbf{Barcodes}$$

With this definition we can define a **decorated bottleneck distance** in terms of interleavings of the underlying merge trees and matchings of the associated barcodes.




## Main Theorem (Hierarchy of Stability Results) and Code

For $\mathbb{R}$-spaces $X_f := f : X \to \mathbb{R}$ and $Y_g := g : Y \to \mathbb{R}$ we have the following

$$d_{MT}(\mathcal{M}_f, \mathcal{M}_g) \leq d_{DB}(\mathcal{BF}_n, \mathcal{BG}_n) \leq d_{DMT}(\tilde{F}_n, \tilde{G}_n) \leq \delta_I(X_f, Y_g)$$

Moreover, we have experimentally verified this refined distinguishing power in practice.

`https://github.com/trneedham/Decorated-Merge-Trees`

Read (lots) more on the arXiv:

`https://arxiv.org/abs/2103.15804`