STUDENT ARTICLE

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Soliton Solutions of Integrable Systems and Hirota's Method

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Abstract

In this paper we investigate a general class of solutions to various partial differential equations known as solitons or stable solitary wave solutions. We introduce necessary background by considering general solutions of the classical wave equation and some of its variants, focusing on features of linearity, non-linearity, dissipation and dispersion. The Korteweg-de Vries (KdV) equation is presented as an iconic non-linear dispersive wave equation that admits soliton solutions. How soliton solutions are approximated motivates an introduction to the Padé approximation, which seeks convergence by expressing a solution as a quotient G/F of polynomials of exponentially decaying functions. The Padé approximation motivates a substitution that decouples the KdV equation into a pair of equations on the polynomials G and F. The decoupled version of the KdV equation is then greatly simplified by introducing a bilinear differentiation operator known as Hirota's D-operator. Another substitution allows Hirota's D-operator to express the KdV equation in a single bilinear form. This final form illustrates how the perturbation method can be used to produce exact soliton and multi-soliton solutions. The generation of multi-solition solutions in an almost additive fashion with this method is summarized as a non-linear superposition principle. Connections between Hirota's method, Kac-Moody algebras and quantum field theory are briefly mentioned.

4.1 Introduction

The study of the dynamical behavior of physical systems has been, and continues to be, a major source of mathematical inspiration. The twentieth century in particular has initiated a deep inquiry into a variety of non-linear systems and their unifying themes. In the spectrum of dynamics, two opposites have attracted considerable attention: chaos and solitons. Chaos theory has demonstrated that both partial and ordinary differential equations can exhibit incredibly rich behavior, allowing some deterministic systems to be exponentially unpredictable for increasing time. On the other extreme, soliton theory provides several important examples of non-linear systems behaving in a stable, quasi-linear fashion.

In this paper, we explore this second extreme and build up a concrete introduction to solitons via an inspection of the Korteweg-de Vries (KdV) equation—a non-linear dispersive equation, which is effective for describing surface waves in a shallow water domain. The existence of stable "solitary waves"—the precursors for the term "soliton"—was first discovered experimentally in 1834 by J. Scott Russell, who chased on horseback a one foot high and 30 feet long wave generated by a stopping canal boat, traveling at eight to nine miles an hour for nearly two miles in unaltered form. This solitary wave solution was re-discovered as a solution to the KdV equation in 1895 [DJ]. Since then, stable solitary wave solutions have featured prominently in many other non-linear

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partial differential equations (PDEs) and the methods for generating soliton solutions have led to many deep ideas in mathematics and physics.

The goal of this paper is to provide an intuition for some of these results. We begin with a simple description and definition of classical linear wave equations in Section 4.2. The one-dimensional wave equation is solved using d'Alembert's method in Section 4.2.1. Explicit plane wave solutions to the wave equation are described in Section 4.2.2 and the relevant terminology of dispersion relations, phase and group velocities are defined in Section 4.2.3. Once this relevant background is covered, we consider in Section 4.2.4 a lesser-known class of solutions to the wave equation that cannot be approximated by plane waves, called solitary waves or solitons.

Pursuant to our objective to understand which PDEs admit soliton solutions, we consider in Section 4.3 generalized wave equations that more accurately model various physical phenomena. In particular, we stress the effects of linearity, non-linearity, dispersion and dissipation on solutions to the corresponding PDEs in Sections 4.3.1, 4.3.2, 4.3.3 and 4.3.4.

Finally, we focus on the aforementioned KdV equation in Section 4.4. We first outline some of the nice properties of the KdV equation and the conservation laws it obeys in Section 4.4.1. It is stated, but not proved, that the KdV equation satisfies infinitely many conservation laws and this relates to its *integrability*. Our focus then returns to solitary wave solutions to the KdV equation. The fact that these solutions cannot be approximated by plane wave solutions leads us to introduce the Padé approximation in Section 4.4.2, which will motivate us to decouple the KdV equation into two equations whose solutions are polynomials of exponentially decaying functions. Padé approximation will lead us to consider in Section 4.4.3 how the perturbation method can approximate solutions to the KdV equation. Our attempt to unite Padé approximation and the perturbation method via a decoupled pair of equations will be our way of motivating and introducing *Hirota's method* in Section 4.4.4. A change of variables suggested by Hirota's method will allow us to put the KdV equation into a very elegant *bilinear* form. Before further exploring this new form, we graphically demonstrate the notion of a *two-soliton solution* in Section 4.4.5, and qualitatively motivate the desire to produce multi-soliton solutions to the KdV equation. The substitution suggested by the Padé approximation is then abandoned in Section 4.4.6 in favor of another change of variables for the KdV equation. This alternative bilinear form will then allow us to apply the perturbation method of Section 4.4.3 to produce exact (not approximate) multi-soliton solutions to the KdV equation in Section 4.4.7. The relationship of this powerful method to deep ideas involving Kac-Moody algebras and quantum field theory are then mentioned briefly in Section 4.5.

Special thanks must be extended to Professor Aliaa Barakat at the Massachusetts Institute of Technology for guiding the author through the rich mathematical vistas involved in integrable systems. Without her direction and support, the current paper would be non-existent.

4.2 The Wave Equation(s)

Definition 1. An equation (or system of equations) is *linear* if, whenever it has solutions u_1 and u_2, it also has $au_1 + bu_2$ as a solution, where a and b are scalar coefficients.

Definition 2. The *classical wave equation* that describes a wave propagating with constant speed c is given by the following linear partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u. \tag{4.1}$$

In one dimension, equation (4.1) models the height of a plucked string as a function of space and time. More specifically, the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{4.2}$$

is an idealized model derived from using force balance and Newton's laws.

4.2.1 d'Alembert's Solution to the 1-D Wave Equation

Putting $\eta = x - ct$ and $\xi = x + ct$ we have that

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial \eta^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \xi^2}, \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \xi^2}.\end{aligned}$$

Substituting into equation (4.2), we find

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0,$$

and integrating twice, we have that solutions take the form

$$u = f(\eta) + g(\xi),$$

where f and g are arbitrary functions. This corresponds to solutions propagating in the left and right directions. If we take equation (4.2) and factor accordingly

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = 0, \tag{4.3}$$

then d'Alembert's solution tells us that if we consider solutions to the simplified wave equation

$$u_t + cu_x = 0, \tag{4.4}$$

we are left with right-traveling solutions only, i.e.

$$u(x,t) = f(\eta) = f(x - ct).$$

4.2.2 Plane Wave Solutions

Definition 3. A *plane wave* is a solution to equation (4.1) that takes the form

$$u(\vec{x},t) = Ae^{i(\vec{k}\cdot\vec{x} - \omega t)} \tag{4.5}$$

where i is the imaginary unit, $\vec{k}$ is the wave vector, ω is the angular frequency, and A is the (possibly complex) amplitude.

For the remainder of this paper, we will restrict our attention to one-dimensional wave equations, in which case k and x are treated as scalar-valued quantities. If x is thought of as having units of length (say meters m), then k must have units that are the corresponding inverse (m^{-1}). It is common to call k the *wave number*.

In the theory of differential equations it is common to "guess" the solution to a given equation by substituting in a function that has certain required properties (most notably it solves the provided differential equation given certain constraints). We call such an assumed form for a solution an *ansatz* for the differential equation. For example, plane waves can be taken as a good ansatz for a solution to many dierent wave equations. If we consider equation (4.4) and assume it has a solution of the form (4.5), $u(x,t) = e^{i(kx-\omega t)}$ then we find that the angular frequency and wave number must satisfy the relation

$$\omega = ck. \tag{4.6}$$

This is an example of a *dispersion relation*.

4.2.3 Dispersion Relations, Phase and Group Velocities

Definition 4. A *dispersion relation* is a relation between the energy of a system and its momentum.

Since energy in waves is proportional to frequency ω and the wave number k is proportional to momentum, equation (4.6) is an example of a dispersion relation. These sorts of relations will be very valuable when considering how fast certain Fourier components in the initial profile of a wave travel and how fast energy dissipates for the given system. One distinguishes between these notions by defining two types of velocities.

To motivate the first definition of velocity, imagine that we are watching an animation of of a one-dimensional wave equation $u(x,t)$ that can be written as function $f(kx-\omega t)$ for k and ω constant. Now imagine we are following a crest of the wave and we notice that at a specific point in time t_0 and point in space x_0, the wave has a height $H = u(x_0,t_0) = f(C_0)$ where $C_0 = kx_0 - \omega t_0$. If we allow a short amount of time Δt to elapse, we see that the point H on the curve has traveled a small distance Δx, so that $u(x_0+\Delta x, t_0+\Delta t) = H$. For Δt and Δx small enough, we see that this can only be the case if

$$kx_0 - \omega t_0 = C_0 = k(x_0+\Delta x) - \omega(t_0+\Delta t),$$

e.g. the point that gets mapped to H is the same point after the wave has traveled a small distance. This is only true if

$$k\Delta x = \omega \Delta t \qquad \Rightarrow \qquad \frac{\Delta x}{\Delta t} = \frac{\omega}{k}.$$

This is known as the *phase velocity* of the wave.

Definition 5. For a wave of the form $u(x,t) = f(kx-\omega t)$ where k and ω are constants, the *phase velocity* c_{ph} is defined as the constant

$$c_{ph} := \frac{\omega}{k}. \tag{4.7}$$

Although the phase velocity can be defined more generally, where c_{ph} determines the speed at which any one frequency component travels, we will restrict ourselves to the definition given above.

Definition 6. The propagation of energy in a system is given by the velocity of wave packets, known as the *group velocity*, which is given by

$$c_{gr} := \frac{\partial \omega}{\partial k}. \tag{4.8}$$

In the case of equation right-traveling waves $f(x-ct)$ (4.4), we determined the linear dispersion relation $\omega = ck$ (4.6). Applying the definitions for the group (4.8) and phase (4.7) velocities, we see that in this instance

$$c_{ph} = \frac{\omega}{k} = \frac{ck}{k} = c = \frac{\partial \omega}{\partial k} = c_{gr}.$$

Definition 7. A *non-dispersive wave* is a wave which is governed by a linear dispersion relation.

For linear equations, differentiation of ω gives the coefficient of k, which is similarly achieved by division. Thus a linear relation implies equality of c_{ph} and c_{gr}. Conversely, imagine that the equation $c_{ph} = c_{gr}$ holds. Treating ω as a function only of k, we can separate variables and solve the differential equation as follows:

$$\begin{aligned} \frac{d\omega}{dk} &= \frac{\omega}{k} \\ \frac{d\omega}{\omega} &= \frac{dk}{k} \\ \log \omega &= \log k + c. \end{aligned}$$

Exponentiating both sides of the equation we obtain the linear dispersion relation $\omega = Ck$ where $C = e^c$. Dispersive waves have unequal phase and group velocities, while non-dispersive waves have equal phase and group velocities.

4.2.4 Solitary Wave Solutions

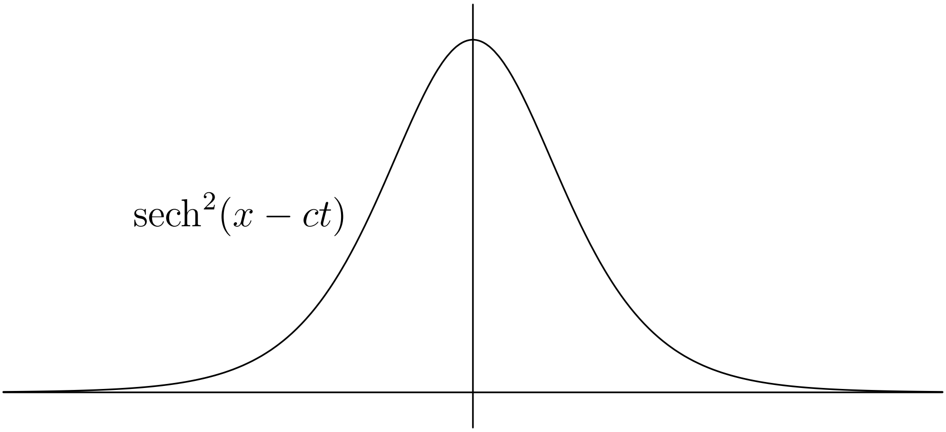

Figure 4.1: Plot of a Solitary Wave Solution

Plane wave solutions are **not** the only solutions to the classical linear wave equations presented in equations (4.1), (4.2), (4.4). For example, Figure 4.1 shows a completely different solution that is not of this plane wave form. Letting

$$u(x,t) = \text{sech}^2(x - ct), \tag{4.9}$$

and calculating

$$\begin{aligned} u_t &= -2c\text{sech}^2(x-ct)\tanh(x-ct), \\ u_x &= 2\text{sech}^2(x-ct)\tanh(x-ct), \end{aligned}$$

we have thus verified that a wave of the form (4.9) satisfies our simplified right-traveling linear wave equation (4.4).

Although not obvious at this point, a solution of the form (4.9) is an example of a *solitary wave solution* or *soliton*.

Definition 8. ([DJ], [ZK]) A *solitary wave solution* or *soliton* is a solution to any wave equation that satisfies the following three properties:

1. retains its shape (initial profile) for all time,

2. is localized (asymptotically constant at $\pm\infty$ or obeys periodicity conditions imposed on the original equation),

3. can pass through other solitons and retain size and shape.

As we will see in the next few sections, other types of wave equations either permit or dismiss the possibility of solitary wave solutions. Of particular interest will be the case when the wave equation under consideration is *non-linear*. Certain non-linear equations will allow solitary wave solutions. In these instances, the third condition in Definition 8 will become especially important as many localized solutions tend to scatter off of one another irreversibly. This is in sharp contrast to linear equations, where two waves can pass through each other without change. Since solitons exhibit at most a phase change after interaction, we will be able to "add" (in a well-defined way to be described later) two soliton solutions to obtain a third one, achieving in effect a *non-linear superposition principle!*

4.3 Generalized Wave Equations

In the previous section, we considered an example of a *linear, non-dispersive* wave equation and the types of solutions it allows. However, these sorts of equations are often inadequate for describing the rich dynamical behavior of the universe. Physical models for vibrations, gravity waves, internal waves, surface waves and a broad range of related phenomena require a mix of dissipative, dispersive and nonlinear behavior. In this section we will consider various combinations of these features and discuss whether or not they support stable solitary wave solutions.

4.3.1 Linear Dispersive Waves

Let us consider a partial differential equation with odd spatial derivatives, such as

$$u_t + c_0 u_x + \delta u_{xxx} = 0, \tag{4.10}$$

where c_0 and δ are constant. Here we are using *subscript-notation* where $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ and so on. Taking as our ansatz the plane wave solution $u(x,t) = e^{i(kx-\omega t)}$, we get the following *non-linear* dispersion relation between the frequency ω and the wave number k

$$\omega = c_0 k - \delta k^3.$$

Applying the definitions for phase and group velocities we find that

$$c_0 - \delta k^2 = \frac{\omega}{k} = c_{ph} \neq c_{gr} = \frac{\partial \omega}{\partial k} = c_0 - 3\delta k^2.$$

If $\delta > 0$ we then have that

$$c_{gr} \leq c_{ph}.$$

If we assume that the Fourier transform of $u(x,0)$—call it $A(k)$—has a continuum of wave numbers in its initial profile, then the evolution of the profile is given by

$$u(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx-\omega(k)t)} dk$$

and the Fourier components literally "disperse" or separate out according to their wave numbers. This example demonstrates that a linear dispersive wave does not exhibit stable solitary wave solutions since an initial profile consisting of many superposed wave numbers breaks apart into individual components instead of retaining its shape.

4.3.2 Linear Dissipative Waves

What is interesting to note is that we get *real* dispersion relations for ω whenever we have *odd order derivatives* for our spatial variable x. If we consider the alternative equation

$$u_t + c_0 u_x - \delta u_{xx} = 0, \tag{4.11}$$

the dispersion relation then is

$$\omega = c_0 k - i\delta k^2.$$

We thus have the solution

$$u(x,t) = e^{-k^2 t + ik(x-t)},$$

which decays exponentially with time.

Definition 9. A *dissipative wave* is a wave whose energy decreases as time increases.

In Section 4.2.3 we noted that energy in waves is proportional to frequency. This statement is true for a given amplitude, but energy is also proportional to a wave's amplitude. Although energy is a difficult concept to define generally, for a wave periodic in x with period T we may define the energy E of a wave $u(x,t)$ as

$$E \equiv \frac{1}{2}\int_0^T |u(x,t)|^2 dx.$$

For the above example, we find that

$$E \propto e^{-2k^2 t},$$

which clearly goes to zero as $t \to \infty$.

4.3.3 Non-Linear Non-Dispersive Waves

A common feature of equations (4.1), (4.10), and (4.11) is their *linearity*. With such systems, once two solutions are produced their sum is guaranteed to be a solution, and we can find a basis for the space of all solutions—the tools of linear algebra are at our whim. *Non*-linear systems are much harder to study precisely for this reason. Let us consider a simple example where the wave's speed c depends on its amplitude. The equation

$$u_t + c(u)u_x = 0, \tag{4.12}$$

where $c(u) = c_0 + bu^n$ for b a constant is one such example. It is clearly non-linear, because if we consider two solutions u and v and we substitute their sum into equation (4.12), we get that

$$u_t + v_t + c_0 + b(u+v)^n(u_x + v_x) = 0$$

if and only if $(u+v)^n = 0$, which is not true in general.

Surprisingly, the solution to equation (4.12) follows (almost) precisely d' Alembert's solution, except that c in the solution $u(x,t) = f(x-ct)$ for equation (4.4) is replaced by the function $c(u)$. This has important implications, because if $c(u)$ is increasing, then the wave travels faster as its amplitude increases, until finally the wave steepens and breaks (where "breaking" in mathematical terms is multi-valuedness).

4.3.4 Non-Linear Dispersive Waves

The upshot of the previous few subsections has been that neither linear dispersive nor non-linear non-dispersive wave equations admit solitary wave solutions. In those cases, a wave profile either has the tendency to break up according to wave number (dispersive) or steepen to multi-valuedness (non-linearity). One might imagine that a mix of these two behaviors would yield even more wild behavior, but it is surprising that these two effects can actually neutralize each other to produce soliton solutions.

We wish to provide an intuitive argument that a wave equation of the form

$$u_t + c_0 u_x + bu^n u_x + \delta u_{xxx} = 0 \tag{4.13}$$

has a solitary wave solution that retains its shape. If we suppose that equation (4.13) has a solitary

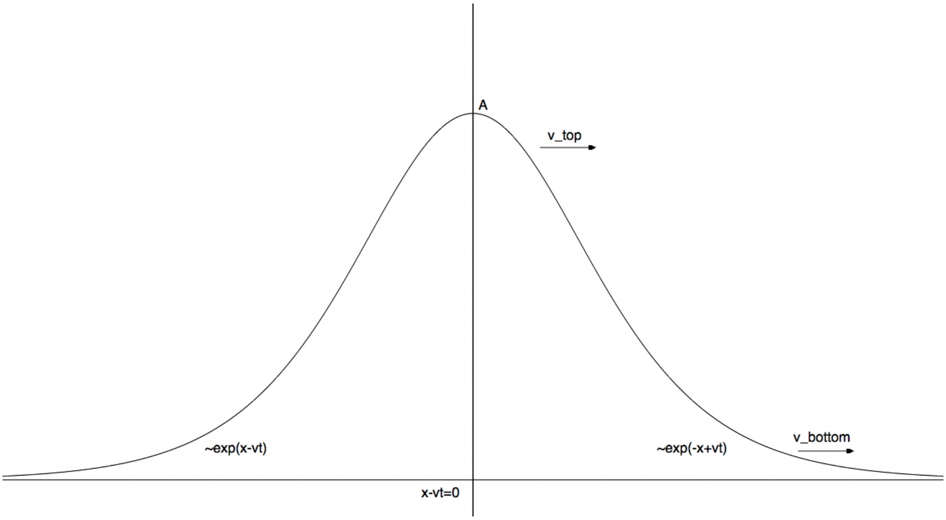

Figure 4.2: Solitary Wave Solution: Intuitive Argument

wave solution as depicted in Figure 4.2, then a necessary assumption for the wave to retain its shape is the condition

$$v_{\text{top}} = v_{\text{bottom}} = v,$$

where v_{top} and v_{bottom} denote the speed at the top and bottom respectively. If we move to a coordinate system that travels with the wave we can hopefully simplify the analysis. Putting $\chi = \ell x - \Omega t$ where $v = \Omega/\ell$, so that at $\chi \sim x$ $t = 0$ and for $\chi \sim x - vt$ for all other times, we are left with a problem in terms of χ.

If $u_{\max} = A$, the amplitude of the wave, then we can approximate our solution in a neighborhood of the top as

$$u \sim A(1 - \text{const} \times \chi^2).$$

We then have that $u_{xxx} \sim 0$ in this neighborhood, so equation (4.13) reduces to exactly the equation we had in Section 4.3.3:

$$u_t + (c_0 + bu^n)u_x \sim 0.$$

Accordingly,

$$v_{\text{top}} = c_0 + bA^n,$$

and if $b > 0$, then $v_{\text{top}} > c_0$.

At the bottom of the wave $u^n \sim 0$, and thus we can neglect the non-linear term, reducing equation (4.13) to

$$u_t + c_0 u_x + \delta u_{xxx} \sim 0. \tag{4.14}$$

Since this is exactly equation (4.10), we know that

$$c_{ph} = c_0 - \delta k^2 = v_{\text{bottom}}, \tag{4.15}$$

and if $\delta > 0$, then $v_{\text{bottom}} < c_0$. This would lead us to conclude that the velocity at the top of the wave is larger than the velocity at the bottom, and thus our wave will steepen and break. But this contradicts the stability of a solitary wave solution! Where did we go wrong in our analysis? The answer is that the expression (4.15) assumes that at the bottom u has a plane wave form! If instead we use $e^{\pm\chi}$ as our anstaz for (4.10), we derive the following non-linear dispersion relation

$$\Omega = c_0\ell + \delta\ell^3,$$

which in turn implies

$$v_{\text{bottom}} = \frac{\Omega}{\ell} = c_0 + \delta\ell^2.$$

We now have the important result that

$$v_{\text{top}} = v_{\text{bottom}} \qquad \Leftrightarrow \qquad \delta\ell^2 = bA^n.$$

The above argument demonstrates several important lessons:

- A non-linear dispersive wave equation of the form (4.13) has a solitary wave solution which moves at constant speed v while preserving its shape.
- A solitary wave solution to a non-linear wave equation cannot be approximated by a plane wave solution $u \sim e^{i(kx-\omega t)}$, but rather requires an exponentially decaying solution of the form $u \sim e^{\pm(\ell x - \Omega t)}$ where $v = \Omega/\ell$.

Later in this paper we will expand on the details for the second point when we introduce a perturbation method for generating soliton solutions. To further motivate and focus our discussion we will restrict our attention to a historically important example of a non-linear dispersive wave equation, which admits soliton solutions, known as the *KdV equation*. After introducing the KdV equation and illustrating some of its properties, we will dive deeper into just how it produces soliton solutions.

4.4 The Korteweg-de Vries (KdV) Equation

One of the first PDEs for which soliton solutions were discovered is the Korteweg-de Vries (KdV) equation,

$$u_t + 6uu_x + u_{xxx} = 0.$$

As described in the introduction, this equation is useful for describing surface waves in a shallow water domain. It is straightforward to verify that

$$u(x,t) = \frac{\ell^2}{2}\operatorname{sech}^2\left(\frac{\chi}{2}\right), \qquad \chi = \ell x - \Omega t, \qquad \Omega = \ell^3, \tag{4.16}$$

is a traveling wave solution to the KdV equation.

4.4.1 Conservation Laws

One of the nice features of the KdV equation (and deeply connected to its integrability) is that it admits infinitely many conservation laws. The two bottom rungs on this conservation ladder are conservation of mass and energy. If we require the KdV equation to obey the periodicity condition

$$u(x+1,t) = u(x,t),$$

then we can prove that the "mass"

$$M := \int_0^1 u(x,t)dx$$

and the "energy"

$$E := \int_0^1 \frac{1}{2}u^2(x,t)dx$$

are independent of time. Simply differentiating with respect to time we have

$$\begin{aligned}\frac{dM}{dt} &= \int_0^1 u_t \\ &= \int_0^1 -6uu_x - u_{xxx} \quad \text{(KdV)} \\ &= [-3u^2]_0^1 - [u_{xx}]_0^1 \\ &= 0\end{aligned}$$

if u and u_{xx} are assumed to be periodic in x.

Using the same conditions on u, we see that

$$\begin{aligned}\frac{dE}{dt} &= \int_0^1 uu_t \\ &= -\int_0^1 6u^2u_x - \int_0^1 uu_{xxx} \quad \text{(KdV)} \\ &= -\int_0^1 \frac{\partial}{\partial x}(2u^3) - \int_0^1 \frac{\partial}{\partial x}(uu_{xx}) + \int_0^1 u_x u_{xx} \\ &= [\frac{1}{2}u_x^2]_0^1 \\ &= 0.\end{aligned}$$

Both M and E are also independent of time if we do not enforce periodicity, but rather require $u, u_x, u_{xx} \to 0$ as $x \to \pm\infty$ and where M and E are integrated on the whole real line. However, this is the case with square-integrable functions, so the demand is not too strict.

4.4.2 Padé Approximation

Although the infinity of conservation laws already points to some of the deeper aspects of the KdV equation, we are primarily concerned with how the KdV equation generates soliton solutions. As detailed in an earlier section, solitary wave solutions cannot be approximated by plane wave solutions and instead require exponentially decaying solutions of the form $e^{\pm\chi}$, where $\chi = \ell x - \Omega t$. In particular, we need to expand u in terms of $\epsilon\exp(\chi)$ where ϵ is small. Unfortunately, the right-hand side in the expression

$$u(x,t) \sim \epsilon a_1 \exp(\chi) + \epsilon^2 a_2 \exp(2\chi) + \cdots \tag{4.17}$$

may diverge for large χ, contrary to the behavior required by a solitary wave solution. Indeed, as suggested by Figure 4.2, we expect that

$$\text{as} \quad \chi \to +\infty, \qquad u(x,t) \sim \exp(-\chi).$$

One way to achieve convergence is to find the *Padé approximation* $u = G/F$ of (4.17), where G and F are polynomials in $\exp(\chi)$.

Definition 10. Given that a function $f(x)$ is $m+n$ times differentiable, the *Padé approximant* of order (m,n) is the rational function

$$R(x) = \frac{p_0 + p_1x + p_2x^2 + \cdots + p_m x^m}{q_0 + q_1x + q_2x^2 + \cdots + q_n x^n} = \frac{G[x]}{F[x]}$$

which agrees with $f(x)$ to the highest possible order, i.e.

$$\begin{aligned} f(0) &= R(0) \\ f'(0) &= R'(0) \\ &\vdots \\ f^{m+n}(0) &= R^{m+n}(0) \end{aligned}$$

Example 11. Suppose

$$f(x) = x - x^3 + x^5 - x^7 + \cdots$$

which converges for $|x| < 1$. Factoring, we obtain an alternating geometric series in terms of x^2

$$x(1 - (x^2)^1 + (x^2)^2 - (x^2)^3 + \cdots) = \frac{x}{1 + x^2} \tag{4.18}$$

and thus $f(x) \sim x^{-1}$ as $x \to \infty$.

Substituting $\exp(\chi)$ for x in (4.18) gives

$$\begin{aligned} f(\exp(\chi)) &= \exp(\chi) - \exp(3\chi) + \exp(5\chi) - \cdots \\ &= \frac{\exp(\chi)}{1 + \exp(2\chi)} \sim \exp(-\chi) \quad \text{as} \quad \chi \to \infty. \end{aligned}$$

4.4.3 Perturbation Method

Considering again the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{4.19}$$

and applying the *perturbation method* one expands u as a power series in a small parameter ϵ to obtain an infinite sequence of *linear* equations on the components of the expansion as follows. We substitute the following expression for u in (4.19)

$$u = \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots, \tag{4.20}$$

and by collecting like powers of ϵ we obtain the following series of equations:

$$\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) u_1 = 0, \tag{4.21}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) u_2 = -6u_1 \frac{\partial u_1}{\partial x}, \tag{4.22}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) u_3 = -6\left(u_2 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_2}{\partial x}\right),$$

$$\vdots$$

As explained in Section 4.13, we need to choose an exponentially decaying solution for u. Let

$$u_1 = a_1 \exp(\chi), \quad \chi = \ell x - \Omega t, \quad \Omega = \ell^3,$$

where a_1 and ℓ are arbitrary. Substituting this solution into (4.22), we determine that

$$u_2 = a_2 \exp(2\chi), \quad a_2 = \frac{-a_1^2}{\ell^2}.$$

Proceeding successively, we can thus find all the u_i's in (4.20), obtaining a power series in $\epsilon\exp(\chi)$ as in (4.17). However, as already mentioned, this expression will diverge for large χ, which we can try to circumvent via a Padé approximation. The trouble with this approach is that there is no simple way to determine the required functions G and F. One potential trick is to reverse engineer a known solitary wave solution so that it takes the form G/F. In the case of (4.19) the one-soliton solution (4.16) is

$$\begin{aligned} u(x,t) &= \frac{\ell^2}{2}\operatorname{sech}^2\left(\frac{\chi}{2}\right), \\ &= \frac{\ell^2}{1+\cosh(\chi)}, \\ &= \frac{2\ell^2}{2+\exp(\chi)+\exp(-\chi)}, \end{aligned}$$

so

$$\begin{aligned} \frac{G}{F} &= \frac{2\ell^2\exp(\chi)}{1+2\exp(\chi)+\exp(2\chi)} \\ &= \frac{2\ell^2\exp(\chi)}{(1+\exp(\chi))^2}. \end{aligned} \tag{4.23}$$

The problem with reverse engineering is that (4.23) is an artificial byproduct of a solution we already know. What would be better is to develop a method which determines the functions G and F without a priori having the solution u. This approach is embodied by Hirota's method.

4.4.4 Hirota's Method

Our practice with the perturbation method and Padé approximations suggests that making the substitution $u(x,t) = G[\exp(\ell x - \Omega t)]/F[\exp(\ell x - \Omega t)]$ in (4.19) to obtain equations for G and F may be a fruitful undertaking. We first calculate:

$$\begin{aligned} u &= \frac{G}{F} \\ u_t &= \frac{G_tF - GF_t}{F^2} \\ u_x &= \frac{G_xF - GF_x}{F^2} \\ u_{xxx} &= \frac{G_{xxx}}{F} - \frac{3G_{xx}F_x + 3G_xF_{xx} + GF_{xxx}}{F^2} \\ &\quad +6\frac{G_xF_x^2 + GF_{xx}F_x}{F^3} - \frac{GF_x^3}{F^4}. \end{aligned}$$

Substituting into the KdV equation (4.19), we obtain the following complicated equation

$$\begin{aligned} u_t + 6uu_x + u_{xxx} &= \frac{G_tF - GF_t}{F^2} + 6\frac{G}{F}\frac{G_xF - GF_x}{F^2} \\ &\quad + \frac{G_{xxx}F - 3G_{xx}F_x - 3G_xF_{xx} - GF_{xxx}}{F^2} \\ &\quad +6\frac{FG_xF_x^2 + FGF_{xx}F_x - GF_x^3}{F^4} \\ &= 0 \end{aligned} \tag{4.24}$$

At first glance equation (4.24) has only made things worse. We could try to decouple this equation into a simpler set of equations. Re-expressing the $6uu_x$ term with F^3 in the denominator as one with F^4 in the denominator, we could require that individually the term with F^2 in the denominator and F^4 in the denominator are both zero:

$$G_tF - GF_t + G_{xx}F - 3G_{xx}F_x - 3G_xF_{xx} - GF_{xxx} = 0 \quad (4.25)$$
$$GG_xF^2 - G^2FF_x + G_xFF_x^2 + GFF_xF_{xx} - F_x^3G = 0. \quad (4.26)$$

Unfortunately, the G and F we derived in (4.23) do not satisfy (4.25) and (4.26), but only by a missing factor of $6G_xF_{xx}$. Changing the minus sign in front of $3G_xF_{xx}$ to a plus sign and transferring the remainder to the numerator of the F^4 term, the KdV equation (4.19) becomes

$$\frac{G_tF - GF_t + G_{xx}F - 3G_{xx}F_x + 3G_xF_{xx} - GF_{xxx}}{F^2} + 6(G_xF - GF_x)\frac{GF - (FF_{xx} - F_x^2)}{F^4} = 0.$$

Setting the terms with denominators F^2 and F^4 equal to zero, we obtain the decoupled equations:

$$G_tF - GF_t + G_{xx}F - 3G_{xx}F_x + 3G_xF_{xx} - GF_{xxx} = 0 \quad (4.27)$$
$$GF - (FF_{xx} - F_x^2) = 0. \quad (4.28)$$

We have done a great deal of work, but it doesn't appear to have paid off. However, careful analysis of the pattern of derivatives suggests that (4.27) and (4.28) can be written even more simply. Introducing a new *bilinear differentiation operator*, Hirota's D-operator, will greatly simplify these expressions once and for all.

Definition 12. The *Hirota D-operator* for two n-times differentiable functions f and g is defined by:

$$D_x^n f \cdot g = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)|_{x_1=x_2=x} \quad (4.29)$$

Example 13. We determine the following quantities

$$\begin{aligned} D_t f \cdot g &= f_t g - f g_t, \\ D_x f \cdot g &= f_x g - f g_x, \\ D_x^2 f \cdot g &= f_{xx} g - 2 f_x g_x + g_{xx} f, \\ D_x^2 f \cdot f &= 2 f_{xx} f - 2 f_x^2, \\ D_x^3 f \cdot g &= f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{xxx}. \end{aligned}$$

With the Hirota D-operator in hand, we then immediately recognize that equations (4.27) and (4.28) for G and F reduce to the following *quadratic*, also called *bilinear*, form:

$$(D_t + D_x^3)G \cdot F = 0, \quad (4.30)$$
$$2GF - D_x^2 F \cdot F = 0. \quad (4.31)$$

Equations (4.30) and (4.31) are the culmination of this section. Not only is the form aesthetically pleasing, but we will soon see how such a form enables one to produce soliton solutions in an almost trivial manner. Before this method for producing such solutions is presented, we would like to first understand what it means for the KdV equation to admit a *multi-soliton*, or *N-soliton*, solution.

4.4.5 N-Soliton Solutions

In section 4.2.4 we outlined the defining characteristics of a soliton solution. In particular, we noted how the third condition in Definition 8 gives rise to a non-linear superposition principle. So far, we

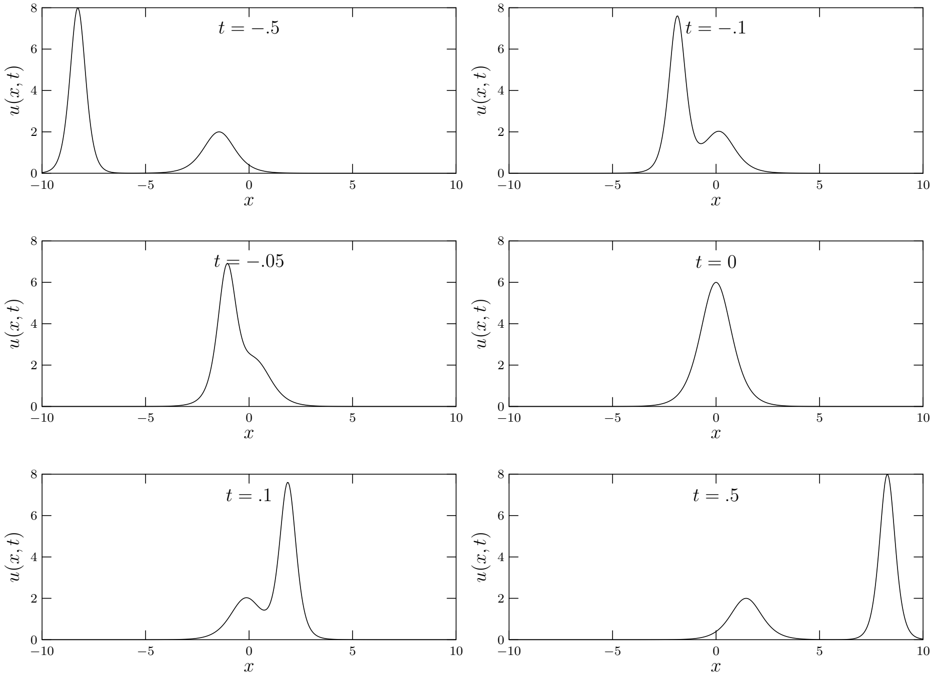

Figure 4.3: Two Soliton Solution for Various Times

did nothing to illustrate this principle graphically, nor did we explain how an equation might admit multiple soliton solutions. In Figure 4.3 we have graphed the following *two-soliton* solution to the KdV equation (4.19) for six different times (p.75 [DJ]):

$$u(x,t) = 12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}. \tag{4.32}$$

One of the striking features of Figure 4.3 is that the tall wave actually catches up to, and passes right through, the smaller wave in an almost *linear* fashion. Careful inspection and exploration by the reader will reveal that after the interaction, the short wave has actually been pushed back and the tall wave has advanced forward relative to where they would have been if the waves had evolved individually, without interaction. This *phase-shift* is the trademark of the non-linearity of the KdV equation. Aside from this small difference, the ability for individual solitary waves to interact strongly and retain their shape is the defining characteristic of solitons. How multiple soliton solutions such as (4.32) are produced for the KdV equation in a more direct, algebraic fashion is the objective of the final few sections. In particular, we will find that reformulating the KdV equation into another bilinear form will allow us to simplify our analysis considerably.

4.4.6 Logarithmic Substitution for the KdV Equation

We motivated the decoupling of the KdV equation into two equations involving G and F by introducing the Padé approximant and asking for a better method to produce these polynomials in $\exp(\chi)$. We could conceivably try to express G and F in terms of a power series like (4.20), substituting into the two bilinear equations (4.30) and (4.31) and obtaining *pairs* of equations analogous to equations (4.21), (4.22), and so on. This, however, turns out to be a rather complicated approach as it stands, so in this section we will introduce an alternative substitution that reduces the KdV equation to another *single* equation in bilinear form. This will allow us to apply the perturbation method to produce *exact* multi-soliton solutions.

Instead of taking $u = G/F$, let us make the substitution

$$u = 2\frac{\partial^2}{\partial x^2}\log f.$$

Then (4.19) can be written as

$$2(\log f)_{xxt} + 3\partial_x(u^2) + u_{xxx} = 0,$$

which upon integrating once by x becomes

$$2(\log f)_{xt} + 3u^2 + u_{xx} = 0. \tag{4.33}$$

We now calculate the relevant quantities:

$$\begin{aligned}
u^2 &= 4\left(\frac{f_x}{f}\right)^4 + 4\left(\frac{f_{xx}}{f}\right)^2 - 8\frac{f_x^2 f_{xx}}{f^3},\\
u_{xx} &= -12\left(\frac{f_x}{f}\right)^4 + 24\frac{f_x^2 f_{xx}}{f^3} - 6\left(\frac{f_{xx}}{f}\right)^2 - 8\frac{f_x f_{xxx}}{f^2} + 2\frac{f_{xxxx}}{f},\\
2(\log f)_{xt} &= -2\frac{f_x f_t}{f^2} + 2\frac{f_{xt}}{f}.
\end{aligned}$$

After substituting into (4.33), simplifying and multiplying by $f^2/2$, we obtain the equation

$$ff_{xt} - f_x f_t + 3f_{xx}^2 - 4f_x f_{xxx} + ff_{xxxx} = 0. \tag{4.34}$$

We are now ready to put the KdV equation into an alternate bilinear form involving the Hirota D-operator defined in equation (4.29). In example 13 we calculated several quantities using the D-operator. We now add the following two calculations:

$$D_x D_t f\cdot f = \;= 2(f_{tx}f - f_t f_x) \tag{4.35}$$

$$D_x^4 f\cdot f = 2(ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2). \tag{4.36}$$

If we compare equations (4.35) and (4.36) with the transformed KdV equation (4.34) we can immediately deduce the alternate bilinear form

$$D_x(D_t + D_x^3)f\cdot f = 0. \tag{4.37}$$

Compared against (4.30) and (4.31), equation (4.37) is a simpler equation. In particular, we will find that the perturbation method produces multi-soliton solutions in a direct manner from this equation.

4.4.7 Producing N-Solitons via the D-Operator

We are now in a position to reapply the perturbation method of Section 4.4.3. It is important to note that when we try to solve PDEs such as the KdV equation via the perturbation method, we usually have an expansion of infinite order, whose coefficients we must determine successively. Truncating the solution leaves us with only approximate solutions to the original PDE. In contrast, we will find that applying the perturbation method to equations in bilinear form and choosing our early components wisely will force our infinite expansion to truncate at finite order. This will allow us to produce exact, rather than approximate, solutions via an expansion (4.20) of finite order in ϵ.

First, we take the one-soliton solution (4.16) and write for $\ell = 2, \Omega = \ell^3$, the solution in terms of the transformed equation for f:

$$u(x,t) = 2\text{sech}^2(x-4t) = 4\frac{\partial}{\partial x}\left(\frac{e^{2x-8t}}{1+e^{2x-8t}}\right) = 2\frac{\partial^2}{\partial x^2}\log(1+e^{2x-8t}).$$

If for notational convenience we write

$$B[f,g] := D_x(D_t + D_x^3)f \cdot g,$$

we determine that for $f = 1 + e^{2x-8t}$

$$B[f,f] = B[1,1] + B[1, e^{2x-8t}] + B[e^{2x-8t}, 1] + B[e^{2x-8t}, e^{2x-8t}] = 0,$$

which checks that this is indeed a solution to (4.37). We wish to generalize this solution to account for N-soliton solutions.

We assume that, like in Section 4.4.3, f can be expanded in positive powers of ϵ from which we can obtain an infinite sequence of equations on the components of the expansion. More precisely, we write

$$f = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n(x,t),$$

and substitute this expression into (4.37), which upon collecting powers of ϵ becomes

$$\begin{aligned} &B[1,1] + \epsilon(B[1,f_1] + B[f_1,1]) + \epsilon^2(B[1,f_2] + B[f_1,f_1] + B[f_2,1]) + \cdots \\ &+\epsilon^r(\sum_{m=0}^{r} B[f_m, f_{r-m}]) + \cdots = 0. \end{aligned} \quad (4.38)$$

Expression (4.38) then reduces to a series of equations, where each term with common power of ϵ is required to be zero. We see, using (4.35) and (4.36), that the equation for f_1 reduces to

$$\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) f_1 = 0,$$

which we will rewrite using the following notation

$$\hat{D} := \left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right), \qquad D := \hat{D}\frac{\partial}{\partial x}.$$

The first few equations from (4.38) then become

$$\hat{D} f_1 = 0 \quad (4.39)$$
$$2Df_2 = -B[f_1, f_1] \quad (4.40)$$
$$2Df_3 = -B[f_1, f_2] - B[f_2, f_1]. \quad (4.41)$$

We can then easily check that, for $f_1 = \exp(\chi_1)$ where $\chi_i = \ell_i x - \ell_i^3 t + \alpha_i$ for ℓ_i and α_i arbitrary constants,

$$\hat{D} f_1 = 0, \quad B[f_1, f_1] = 0, \quad \text{and} \quad \hat{D} f_2 = 0.$$

Accordingly, we may choose $f_n = 0$ for $n = 2, 3, ...$ in expression (4.38) and we regain the solitary wave solution.

It is at this point that we make the very important observation that equation (4.39) is *linear!* This linearity, as we will now explain, is the key to generating multi-soliton solutions to the KdV equation. Let us assume that

$$f_1 = \exp(\chi_1) + \exp(\chi_2) \quad (4.42)$$

where χ_i is defined above. Since (4.39) is linear we know $\hat{D}f_1 = 0$. For (4.40) we have that

$$\begin{aligned} 2Df_2 &= -B[f_1, f_1] \\ &= -B[\exp(\chi_1), \exp(\chi_1)] - B[\exp(\chi_1), \exp(\chi_2)] \\ &\quad -B[\exp(\chi_2), \exp(\chi_1)] - B[\exp(\chi_2), \exp(\chi_2)]. \end{aligned}$$

Noting the fact that only terms involving both χ_1 and χ_2 are non-zero, we find that

$$2Df_2 = -2\{(\ell_1 - \ell_2)(\ell_2^3 - \ell_1^3) + (\ell_1 - \ell_2)^4\}\exp(\chi_1 + \chi_2). \tag{4.43}$$

Equation (4.43) has a solution of the form

$$f_2 = A_2 \exp(\chi_1 + \chi_2),$$

and upon substituting into (4.43), we find that

$$A_2 = \left(\frac{\ell_1 - \ell_2}{\ell_1 + \ell_2}\right)^2.$$

Proceeding to equation (4.41), we substitute our expressions for f_2 and f_1 and determine that

$$\begin{aligned} 2Df_3 &= -A_2B[\exp(\chi_1), \exp(\chi_1 + \chi_2)] - A_2B[\exp(\chi_1 + \chi_2), \exp(\chi_1)] \\ &\quad -A_2B[\exp(\chi_2), \exp(\chi_1 + \chi_2)] - A_2B[\exp(\chi_1 + \chi_2), \exp(\chi_2)] \\ &= -2A_2\{(-\ell_2)\ell_2^3 + (-\ell_2)^4\}\exp(2\chi_1 + \chi_2) \\ &\quad -2A_2\{(-\ell_1)\ell_1^3 + (-\ell_1)^4\}\exp(2\chi_1 + \chi_2) \\ &= 0. \end{aligned}$$

Notice that in contrast to our one-soliton solution, assuming f_1 has the form (4.42) allows us to truncate (4.38) by putting $f_n = 0$ for $n \geq 3$. Putting $\epsilon = 1$, we now have an *exact two-soliton solution* to the KdV equation:

$$f = 1 + \exp(\chi_1) + \exp(\chi_2) + \left(\frac{\ell_1 - \ell_2}{\ell_1 + \ell_2}\right)^2 \exp(\chi_1 + \chi_2).$$

The method developed above generalizes to any exact N-soliton solution simply by putting

$$f_1 = \sum_{i=1}^{N} \exp(\chi_i) \tag{4.44}$$

and the expansion (4.38) is guaranteed to terminate after the f_N term. Although this termination can be proven, we will not do so here. It is important to note that it is expression (4.44) and its corresponding exact solution that gives us a *non-linear superposition principle*. The ability to take one-soliton solutions and combine them to form multi-soliton turns out to be an important feature of integrable systems. We could, in fact, take this as a definition of integrability.

Definition 14. ([Hi] p. 101) A set of equations written in Hirota bilinear form is *Hirota integrable*, if one can combine any number N of one-soliton solutions into an N-soliton solution.

The fact that (4.44) generates an N-soliton solution to the KdV equation (4.37) is testament to its Hirota integrability. In all cases known so far, Hirota integrability has turned out to be equivalent to more conventional definitions of integrability [Hi].

Although other methods for finding exact multi-soliton solutions exist, the Hirota D-operator is considered to be the most direct and algebraic method for doing so. The geometric and deeper connections of Hirota's method with some other areas of mathematics and physics will be discussed briefly in the next section.

4.5 Directions Forward

The field of integrable systems has seen many spectacular developments in the past several decades. This growth can largely be attributed to the fruitful exchange that occurs at the nexus of mathematics and physics—a nexus occupied by the study of Riemann surfaces, Kac-Moody algebras, twistor theory and quantum field theory. It is our hope that in this paper we have illustrated, starting with simple physical phenomenon, some of the beautiful mathematical structure that lies behind our models of the universe.

Historically speaking, Hirota's method was discovered by some of the very same brute-force calculations and by-hand manipulations carried out here [Ba]. It was only later, through the work of the Japanese mathematicians Date, Jimbo, Miwa, Kashiwara, Sato and Sato, that the deep connections between Hirota's bilinear form for non-linear PDEs and Kac-Moody algebras were discovered.

The seemingly arbitrary substitutions made to reduce the KdV equation to its bilinear form, either the rational substitution $u = G/F$ or the logarithmic substitution $u = 2(\log f)_{xx}$, are actually part of a broader class of functions known as τ-functions. The discovery that the partition functions of several important quantum field theories are τ-functions of non-linear PDEs is a major theme of current research in theoretical physics [Ba].

References

[Ba] Aliaa Barakat. Personal Discussions with the Author, 01/2008.

[DJ] P. G. Drazin and R. S. Johnson. *Solitons: an introduction*. Cambridge University Press, 1989.

[Hi] J. Hietarinta. *Introduction to the Hirota Bilinear Method*, volume 638 of *Lect. Notes Phys.* Springer-Verlag, 2004.

[ZK] N. J. Zabusky and M. D. Kruskal. Interaction of "Solitons" in a Collisionless Plasma and the Recurrence of Initial States. *Phys. Rev. Lett.*, 15(6):240–243, Aug 1965.